By combining Eqs. 8.79 and 8.80, we have

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\frac{\exp[-\int_a^x P(x_1)\,dx_1]}{y_1^2}. \tag{8.81}$$

Finally, by integrating Eq. 8.81 from $x_2 = b$ to $x_2 = x$ we get

$$y_2(x) = y_1(x)W(a)\int_a^x \frac{\exp[-\int_a^{x_2} P(x_1)\,dx_1]}{[y_1(x_2)]^2}\,dx_2. \tag{8.82}$$

Here a and b are arbitrary constants and a term $y_1(x)\ y_2(b)/y_1(b)$ has been dropped, for it leads to nothing new. Since $W(a)$, the Wronskian evaluated at $x = a$, is a constant and our solutions for the homogeneous differential equation always contain an unknown normalizing factor, we set $W(a) = 1$ and write

$$y_2(x) = y_1(x)\int^x \frac{\exp[-\int^{x_2} P(x_1)\,dx_1]}{[y_1(x_2)]^2}\,dx_2. \tag{8.83}$$

Note that the lower limits $x_1 = a$ and $x_2 = b$ have been omitted. If they are retained, they simply make a contribution equal to a constant times the known first solution, $y_1(x)$, hence add nothing new.

If we have the important special case of $P(x) = 0$, Eq. 8.83 reduces to

$$y_2(x) = y_1(x)\int^x \frac{dx_2}{[y_1(x_2)]^2}. \tag{8.84}$$

This means that by using either Eq. 8.83 or 8.84 we can take one known solution and by integrating can generate a second independent solution of Eq. 8.73. This technique is used in Section 12.10 to generate a second solution of Legendre's differential equation.

EXAMPLE 8.6.3 A Second Solution for the Linear Oscillator Equation

From $d^2y/dx^2 + y = 0$ with $P(x) = 0$ let one solution be $y_1 = \sin x$. By applying Eq. 8.84, we obtain

$$y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2}$$
$$= \sin x(-\cot x) = -\cos x,$$

which is clearly independent (not a linear multiple) of $\sin x$.

Series Form of the Second Solution

Further insight into the nature of the second solution of our differential equation may be obtained by the following sequence of operations:

1. Express $P(x)$ and $Q(x)$ in Eq. 8.73 as

$$P(x) = \sum_{i=-1}^{\infty} p_i x^i, \qquad Q(x) = \sum_{j=-2}^{\infty} q_j x^j. \tag{8.85}$$

The lower limits of the summations are selected to create the strongest possible *regular* singularity (at