The Calculus of Variations

As we saw when we discussed path integrals, the amplitude

\[ -i\hbar \frac{\partial}{\partial \phi(x)} \int \mathcal{D}x(t) \mathcal{D}\phi(x(t)) \phi(0) \phi(T) = \langle \phi(x(T)) \phi(x(0)) \rangle \]

is

\[ \exp \left( \frac{-i S[x]}{\hbar} \right) = \int \mathcal{D}x(t) e^{i S[x(t)]/\hbar} \]

where \( S[x] \) is the classical action

\[ S[x(t)] = \int_0^T dt \left[ \frac{\dot{x}^2}{2m} - V(x(t)) \right] \]

Two paths that differ by \( \delta x(t) \) may wash each other out unless the action \( S \) is stationary,

\[ 0 = \delta S \]

which means that \( \delta S \propto \delta x^2 \).

This is the principle of least action.

In fact, much of classical physics follows from a choice of \( S \) and the rule \( 0 = \delta S \).

Example: (Note \( \delta x = \dot{x} + \delta \dot{x} - \dot{x} = d (\delta x)/dt \).)

\[ \delta S = \int_0^T dt \left[ m \delta x \delta \dot{x} - V' \delta x + O(\delta x^2) \right] \]

\[ = \int_0^T dt - \delta x \left( m \ddot{x} + V' \right) + \left[ m \delta x \right]_0^T, \]

since we dropped \( \delta x^2 \).
If the two paths \( x(t) \) and \( x(t) + \delta x(t) \) both go from \( x(0) \) to \( x(T) \), then

\[
\delta x(0) = \delta x(T) = 0
\]

and the boundary terms vanish. Then

\[
\delta S = - \int_0^T dt \left( m \dddot{x} + V' \right) \delta x
\]

so \( \delta S \propto \delta x^2 \), and not \( \delta x \propto \delta S \), then

\[
0 = \delta S = - \int_0^T dt \left( m \dddot{x} + V' \right) \delta x
\]

and since \( \delta x \) is arbitrary (but small), we get

\[
m \dddot{x} + V' = 0 \quad \text{or} \quad m \dddot{x} = -V' \quad \text{or}
\]

\[
m \dddot{x} = F = ma = \frac{\partial V(x)}{\partial x}
\]

In books on classical mechanics, one often uses generalized coordinates \( q_i(t) \) so that

\[
S = \int_0^T dt \, L(q, \dot{q}, t),
\]

in which \( q \) and \( \dot{q} \) stand for \( q_1, \ldots, q_n \), etc.
Now the action $S$ will be stationary if

$$0 = \delta S = \frac{d}{dt} \left( \frac{\partial S}{\partial \dot{\delta} q_i} \right) + \frac{\partial S}{\partial \delta q_i}.$$ 

Now

$$\delta \dot{\delta} q_i = \dot{q}_i + \dot{\delta} q_i - \dot{\delta} q_i = \frac{d}{dt} \delta q_i,$$

so again integrating by parts, we have

$$\delta S = \int_0^T \left( \delta \dot{\delta} q_i \left( \frac{\partial L}{\partial \delta q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\delta} q_i} \right) + \left[ \frac{\partial L}{\partial \delta q_i} \delta \dot{\delta} q_i \right] \right) dt.$$ 

If all the paths go from $q_i(0)$ to $q_i(T)$, then

$$\delta q_i(0) = \delta q_i(T) = 0 \quad \text{and we have}$$

$$0 = \int_0^T \left( \frac{\partial L}{\partial \delta q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\delta} q_i} \right) dt \quad \text{and} \quad \frac{\partial L}{\partial \delta q_i} = 0.$$ 

The canonical momentum $p_i$ is

$$p_i = \frac{\partial L}{\partial \dot{\delta} q_i} \quad \text{and so}$$

Lagrange's equations imply

$$\dot{p}_i = \frac{\partial L}{\partial q_i}.$$
Usually the Lagrangian \( L(\phi, \dot{\phi}) \) does not involve \( t \) explicitly. In this case, one may define a Hamiltonian \( H \) that is conserved:

\[
H = \sum_{i=1}^{N} p_i \dot{\phi}_i - L
\]

To see that \( H \) vanishes, just take its time derivative:

\[
\dot{H} = \sum_{i=1}^{N} \left( p_i \ddot{\phi}_i + p_i \ddot{\phi}_i - \frac{\partial L}{\partial \dot{\phi}_i} \dot{\phi}_i - \frac{\partial L}{\partial \phi_i} \phi_i \right)
\]

But \( p_i = \frac{\partial L}{\partial \dot{\phi}_i} \) and \( \dot{p}_i = \frac{\partial L}{\partial \phi_i} \), so \( \dot{H} = 0 \).

Now suppose \( L(\phi, \phi_i) \) is a Lagrange density that depends on the fields \( \phi, \phi_1, \ldots, \phi_N \) and their derivatives

\[
\phi_{ij} = \frac{\partial \phi_i}{\partial x_j},
\]

Assume \( \delta \phi_i = 0 \) when any argument \( x_k \to \pm \infty \), so we can drop all surface terms. Then

\[
L = \int d^3x L \quad \text{and} \quad S = \int d^4x \, L = \int d^3x \, L.
\]

The requirement that \( SS \) be quadratic or higher in \( \delta \phi \) gives

\[
0 = \delta S = \int d^4x \left( \frac{\partial L}{\partial \phi_i} \delta \phi_i + \frac{\partial L}{\partial \phi_{ij}} \delta \phi_{ij} \right)
\]
Now \( S \delta \phi_{ij} = (\phi_i + \delta \phi_i)_{ij} - \phi_{ij} = \delta \phi_{ij} \)

where \( G_{ij} = \frac{\partial G}{\partial x} \), as before, so

\[
0 = 8 \mathcal{S} = \int d^4x \; S \phi_i \left( \frac{\partial \mathcal{K}}{\partial \phi_i} - \left( \frac{\partial \mathcal{K}}{\partial \phi_{ij}} \right)_{ij} \right) + \mathcal{S} \mathcal{T} \]

whence the field equations

\[
0 = \frac{\partial \mathcal{K}}{\partial \phi_i} - \frac{1}{2} \left( \frac{\partial \mathcal{K}}{\partial \phi_{ij}} \right)_{ij} .
\]

In this way, one does field theory.
Langrange Multipliers

Suppose one wants to find \( x \) and \( y \) that maximize \( f(x, y) \) subject to the constraint \( g(x, y) = c \), a constant. Set

\[
H(x, y, \lambda) = f(x, y) + \lambda (g(x, y) - c)
\]

and find the \( x, y, \) \& \( \lambda \) that maximize \( H \). Then

\[
0 = \frac{\partial H}{\partial x} = H_x = f_x + \lambda g_x = 0 \quad \text{Solve these three eqs. for } x, y, \lambda.
\]

\[
0 = H_y = f_y + \lambda g_y = 0
\]

\[
0 = H_\lambda = g - c = 0
\]

Why does this work? Well, one could solve for the curve \( y(x) \) that keeps

\[
g(x, y(x)) = c.
\]

Then one can maximize \( f(x, y(x)) \) by setting its derivative equal to zero:

\[
0 = f_x + y' f_y.
\]

To find \( y' \), one sets

\[
0 = g_x + y' g_y, \text{ which gives } y' = -\frac{g_x}{g_y},
\]
So,

\[ 0 = f_x + \frac{g_x}{g_y} f_y \]

\[ 0 = f_x - \frac{f_y}{g_y} g_x \]

\[ \lambda = -\frac{f_y}{g_y} \]

and

\[ 0 = f_y + \frac{1}{g_y} f_x \]

\[ 0 = f_y - \frac{g_y f_x}{g_x} = f_y - \frac{f_x}{g_x} g_x \]

\[ \lambda = -\frac{f_x}{g_x} \]

So,

\[ 0 = f_x - \frac{f_y}{g_y} g_x \]

\[ 0 = f_y - \frac{f_x}{g_x} g_x \]

And the two equations for \( \lambda \) are the same because

\[ 0 = f_x - \frac{g_x}{g_y} f_y \text{ which means } \frac{f_x}{g_y} = \frac{f_y}{g_x} = -\lambda. \]

Suppose \( \rho = \Sigma \rho_n \mid n \rangle \langle n \mid \) is a density operator. Then \( \langle 1 \rangle = tr(p F) \)
and \( I = \langle i \rangle = \lambda p_i \), \( \langle H \rangle = \lambda p_i H_i \).

Now \( S = -k \lambda (p \log p) \). So let's maximize \( S \) subject to the conditions \( I = \lambda p \) and \( E = \langle H \rangle \).

So we maximize

\[
Z(p, \lambda, \mu) = S + \lambda (E - \langle H \rangle) + \mu (1 - \langle i \rangle)
\]

\[
= -k \lambda (p \log p) + \lambda (E - \lambda p_i H_i) + \mu (1 - \lambda \lambda p)
\]

We suppose \( \langle n \rangle = \Sigma n m' \).

\[
H \langle n \rangle = E \langle m \rangle . \quad \text{So}
\]

\[
Z(p, \lambda, \mu) = -k \sum p_m \log p_m + \lambda (E - \sum p_m E_m) + \mu (1 - \sum p_m)
\]

\[
0 = \frac{\partial Z}{\partial p_m} = -k \log p_m - \lambda E_m - \mu , \quad \text{so}
\]

\[
\log p_m = \frac{-\lambda E_m - \mu}{k}
\]

\[
p_m = \frac{e^{-\lambda E_m / k}}{\sum e^{-\lambda E_m / k}}
\]

Choose \( \mu \) by setting \( p_m = \frac{e^{-\lambda E_m / k}}{\sum e^{-\lambda E_m / k}} \).

Choose \( \lambda = \frac{1}{T} \)

by the rule \( \sum p_m E_m = E \).
Then

\[ p_n = \frac{e^{-\frac{E_n}{kT}}}{\sum e^{-\frac{E_n}{kT}}} \]

is the quantum density operator that maximize entropy for a fixed mean value \( E \) of the energy while conserving probability \( \sum p = 1 \).
Chaos

Henri Poincaré (~1900) studied the three-body problem and found very complicated (chaotic) orbits.

There seem to be four kinds of classical motion:
1. periodic
2. steady (or damped motion that stops)
3. quasi-periodic (mixture of periodic motions, etc.)
4. chaotic

In a system after a transient period.

Examples:

\[ x'' + v x + x^3 - x = g \sin \omega t \]

Ex. 8 theory

give something like
Dripping faucet

Data are $t_1, t_2, t_3, \ldots$

At low flow rate, $\Delta t_n = t_{n+1} - t_n$ is constant, all $\Delta t_n$ are equal.

At a slightly higher rate, the dips come with gaps that alternate $\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \ldots$ so that $\Delta t_{n+2} = \Delta t_n$. This is a period-two sequence.

At still higher flow rates, no regularity is apparent.
Chaotic Rayleigh–Bénard convection occurs when a fluid is placed in a gravitational field between two plates that are kept at constant temperatures with the lower plate hotter by $\Delta t$ above the chaotic threshold. For lower $\Delta t$, the motion is steady convective cellular flow.

Dynamical Systems

$$\dot{x}_i = F_i(x) \quad \text{and} \quad \dot{x} = F(x),$$

$$\dot{x}(t) = F(x(t))$$

The crossings of a suitably oriented plane give rise to a map

$$x_{n+1} = M(x_n)$$

in some lower dimensions.
In the system

\[ \dot{x} = F(x) \]

chaos can occur only if the dimension \( N \) of the vector \( x \) exceeds 2

\[ N > 3. \]

For the invertible map

\[ x_{n+1} = M(x_n) \Rightarrow x_n = M^{-1}(x_{n+1}) \]

chaos occurs only if \( N > 2. \)

If the map is not invertible, then chaos can occur even if \( N = 1. \) An example is

\[ x_{n+1} = r x_n (1 - x_n) \]

which is not invertible and does exhibit chaos in increasingly striking forms as \( r \) exceeds a number slightly greater than \( R \approx 3.57. \)

By \( r = 4, \) the map is totally chaotic.
Here $x_1 = x_2 = 0$ is an attractor.

The limit cycle occurs in the van der Pol equation

$$\ddot{y} + (\eta^2 - \gamma) \dot{y} + \omega^2 y = 0$$

which may be written as the first-order system

$$\begin{align*}
x_1 &= \dot{y} \\
x_2 &= y
\end{align*}$$

$$\begin{align*}
\dot{x}_1 &= -\omega^2 x_2 - (x_2^2 - \gamma) x_1 \\
\dot{x}_2 &= x_1
\end{align*}$$

The van der Pol equation was introduced in the 1920s to describe a vacuum-tube oscillator circuit.
Fractals

Fractal sets don't have dimensions that are natural numbers. To compute their dimensions one needs a definition of dimension.

The box-counting dimension is as follows: cover the set with line segments, squares, cubes, etc., of edge length \( \epsilon \). Count how many you need as \( \epsilon \to 0 \). Call the number of boxes \( N(\epsilon) \). Then

\[
D_0 = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln (1/\epsilon)}.
\]

Cantor set: 0 \( \overline{0} \)

\[
E_n = \left(\frac{1}{3}\right)^n \quad \text{need} \quad N(\epsilon) = 2^n \quad \text{boxes}.
\]

So

\[
D_0 = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \lim_{n \to \infty} \frac{\ln 2}{\ln 3} = 0.63.
\]

Attractors of fractal dimension are strange.