The function $\frac{1}{|\vec{R}-\vec{r}|}$ occurs throughout electrodynamics and gravity theory. If $R>r$, then we may expand it as a power series in r/R:

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{(R^2+r^2-2\vec{R}\cdot\vec{r})^{\frac{1}{2}}} = \frac{1}{(R^2+r^2-2Rr\cos\theta)^{1/2}}.$$

Letting $x=\cos\theta$ and $t=r/R$, we have

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R(1+t^2-2xt)^{1/2}} = \frac{1}{R}\, g(t,x)$$

where $g(t,x)$ may be expanded in a power series in t for $t<1$

$$g(t,x) = (1+t^2-2xt)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n .$$

Since $t=r/R$, the Green's function is

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{R}\right)^n, \qquad (\vec{r}\cdot\vec{R}=rR\cos\theta)$$

which is often written as

$$\frac{1}{|r-r'|} = \frac{1}{r_>}\sum_{n=0}^{\infty}\left(\frac{r_<}{r_>}\right)^n P_n(\cos\theta).$$

The function $g(t,x)$ is a generating function for the Legendre polynomials $P_n(x)$.

$$\sum_{n=0}^{\infty} t^n P_n(x) = g(t,x) = (1+t^2-2tx)^{-1/2}$$

Using the binomial theorem of Sec. 5.6, Eq. (5.103), we have

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!} x^n = \sum_{n=0}^{\infty} \binom{m}{n} x^n,$$

which converges for $|x|<1$. If m is an integer, then the series terminates at $n=m$ since $(m-n)! = \pm\infty$ for $n>m$. If m is not an integer, then

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

After some algebra, Ex. (10.1.15), one has

$$(1+t^2-2tx)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}(2xt-t^2)^n$$

where $n!! = n(n-2)(n-4)\cdots$ down to 2 or 1. One has

$P_0(x)=1$, $P_1(x)=x$, $P_2(x) = \frac{3}{2}x^2-\frac{1}{2}$, and

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!} x^{n-2k},$$

where $[5/2]=[2.5]=2$, etc., $[x]$ is the greatest integer not greater than x.

Now

$$g(t,x) = (1+t^2-2tx)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \text{ so}$$

$$\frac{\partial g}{\partial t}(t,x) = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n\, P_n(x)\, t^{n-1}$$

So

$$(x-t)\, g(t,x) = (1-2xt+t^2) \sum_{n=0}^{\infty} n\, P_n(x)\, t^{n-1}$$

$$= (x-t) \sum_{n=0}^{\infty} P_n(x)\, t^n$$

By identifying the coefficients of t^n, one has

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x) \qquad (*)$$

e.g.

$$3x P_1 = 2P_2 + P_0 \qquad \text{so}$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1)$$

Computationally one uses the recurrence relation

$$P_{n+1}(x) = 2x\, P_n(x) - P_{n-1}(x) - [x P_n(x) - P_{n-1}(x)]/(n+1).$$

It's more stable than the three-term recurrence relation $(*)$.

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

or

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n - t\sum_{n=0}^{\infty} P_n(x)\, t^n = 0, \text{ so}$$

setting to zero the coefficient of t^n, we get

$$P_{n+1}'(x) + P_{n-1}'(x) = 2x\, P_n'(x) + P_n(x). \qquad (**)$$

We differentiate Eq. $(*)$ and combine it with $(**)$ to get

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)\, P_n(x), \qquad (+)$$

These last two relations lead to get more:

$$P_{n+1}'(x) = (n+1)\, P_n(x) + x\, P_n'(x)$$

$$P_{n-1}'(x) = -n\, P_n(x) + x\, P_n'(x)$$

$$(1-x^2)\, P_n'(x) = n\, P_{n-1}(x) - n x\, P_n(x)$$

$$(1-x^2)\, P_n'(x) = (n+1)x\, P_n(x) - (n+1)\, P_{n+1}(x)$$

and

$$(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0 \qquad (!)$$

This last equation is Legendre's equation. It is self adjoint:

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)\,P_n(x) = 0.$$

With $x = \cos\theta$ $\qquad 1-x^2 = \sin^2\theta$

$$\frac{d}{d\theta} = \frac{d\cos\theta}{d\theta}\frac{d}{dx} = -\sin\theta\,\frac{d}{dx} \quad \text{so}$$

$$\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

and L's eq is

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta\left(-\frac{1}{\sin\theta}\right)\frac{d}{d\theta}P_n(\cos\theta)\right] + n(n+1)P_n(\cos\theta) = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\,\frac{d}{d\theta}P_n(\cos\theta)\right] + n(n+1)\,P_n(\cos\theta) = 0$$

or

$$\frac{d^2}{d\theta^2}P_n(\cos\theta) + \cot\theta\,\frac{d}{d\theta}P_n(\cos\theta) + n(n+1)\,P_n(\cos\theta) = 0.$$

$$g(t,1) = \frac{1}{(1-2t+t^2)^{\frac{1}{2}}} = \frac{1}{((1-t)^2)^{1/2}} = \frac{1}{1-t}$$

$$= \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

So

$$P_n(1) = 1.$$

$$g(t,-1) = \frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n = \sum_{n=0}^{\infty} P_n(-1)\, t^n,$$

so

$$P_n(-1) = (-1)^n.$$

Similarly but with much more effort one shows that

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

and

$$P_{2n+1}(0) = 0, \qquad \text{(zip)}$$

Parity

$$(1+t^2-2tx)^{-\frac{1}{2}} = g(t,x) = g(t,-x) \quad \text{so}$$

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(-x)\, (-t)^n = g(-t,-x).$$

So

$$P_n(-x) = (-1)^n P_n(x) \quad \text{which implies (zip).}$$

Eqs. (12.38 — 12.39c) of the text show that

$$P_n(\cos\theta) = \sum_{m=0 \text{ or } 1}^{n} a_m \cos m\theta \qquad \text{all } a_m \geq 0$$

So the maximum of $P_n(\cos\theta)$ is at $\theta = 0$

$$|P_n(x)| = |P_n(\cos\theta)| \leq P_n(1) = 1.$$

Orthogonality

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0$$

is a self-adjoint ODE. So its eigenfunctions associated with different eigenvalues are orthogonal. Integrating by parts and dropping boundary terms which vanish with $(1-x^2)$ at $x = \pm 1$, we get

$$\int_{-1}^{1} dx \left\{ P_m(x)\left[(1-x^2)P_n'(x)\right]' - P_n\left[(1-x^2)P_m'(x)\right]' \right\}$$

$$= \int_{-1}^{1} dx \left\{ -P_m'(x)\left[(1-x^2)P_n'(x)\right] + P_n'(x)\left[(1-x^2)P_m'(x)\right] \right\}$$

$$= 0 = \left[m(m+1) - n(n+1)\right] \int_{-1}^{1} P_n(x)P_m(x)\,dx$$

So if $m \neq n$

$$\int_{-1}^{1} dx\, P_n(x)P_m(x) = 0 \qquad \text{or}$$

$$\int_{0}^{\pi} P_n(\cos\theta)\, P_m(\cos\theta)\sin\theta\, d\theta = 0$$

After some algebra (Eqs. (12.44) – (12.48) of text), one has

$$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1} .$$

So

$$\int_{-1}^{1} dx \quad P_n(x) \; P_m(x) = \frac{2}{n+1} \, \delta_{nm} .$$

Suppose we expand on $[-1, 1]$

$$f(x) = \sum_{m=0}^{\infty} a_m \, P_m(x) ,$$

Then

$$\int_{-1}^{1} dx \quad P_n(x) \, f(x) = \sum_{m=0}^{\infty} a_m \int_{-1}^{1} dx \, P_n(x) P_m(x)$$

$$= \sum_{m=0}^{\infty} a_m \frac{2}{n+1} \, \delta_{nm}$$

$$= \frac{2 \, a_n}{n+1} , \text{ or}$$

$$a_n = \frac{n+1}{2} \int_{-1}^{1} dx \;\, P_n(x) \; f(x) , \quad \text{and so}$$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = \sum_{n=0}^{\infty} \frac{n+1}{2} P_n(x) \int_{-1}^{1} dy \, P_n(y) \, f(y) .$$

But this means that

$$f(x) = \int_{-1}^{1} dy\, f(y) \left[\sum_{n=0}^{\infty} \frac{n+1}{2} P_n(x)\, P_n(y) \right]$$

$$= \int_{-1}^{1} dy\, f(y) \quad \delta(x-y)$$

So on $[-1,1]$ and for the space of functions spanned by the $P_n(x)$, the Dirac delta function is

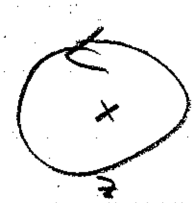
$$\delta(x-y) = \sum_{n=0}^{\infty} \frac{n+1}{2} P_n(x)\, P_n(y).$$

Rodrigues's formula is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n .$$

Schlaefli's integral.

$$(x^2-1)^n = \frac{1}{2\pi i} \oint \frac{(z^2-1)^n}{z-x}\, dz$$



$$\frac{1}{2^n n!} \frac{d\,(x^2-1)^n}{dx^n} = P_n(x) = \frac{1}{2^n 2\pi i} \oint \frac{(z^2-1)^n}{(z-x)^{n+1}}\, dz .$$

The Associated Legendre Functions are simple polynomials in both $\cos\theta$ and $\sin\theta$. They arise as the solutions of the θ-equation that is separated from equations like

$$-\Delta\phi = k^2\phi .$$

The θ-equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP_n^m(\cos\theta)}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]P_n^m(\cos\theta) = 0$$

or with $x = \cos\theta$

$$\left[(1-x^2)\,P_n^{m\prime}(x)\right]' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]P_n^m(x) = 0 \qquad (*)$$

which is self adjoint.

To get $P_n^m(x)$, we use Leibnitz's rule

$$\frac{d^n}{dx^n}AB = \sum_{s=0}^{n}\binom{n}{s}A^{(n-s)}B^{(s)}$$ to differentiate

Legendre's equation

$$\left[(1-x^2)P_n'\right]' + n(n+1)P_n = 0$$

m times, obtaining

$$(1-x^2)\,u'' - 2x(m+1)\,u' + (n-m)(n+m+1)\,u = 0$$

where $u = P_n^{(m)} = \dfrac{d^m}{dx^m}P_n(x)$.

We make this equation self adjoint by setting

$$v(x) = (1-x^2)^{\frac{m}{2}} u(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x),$$

whence, after some work, we get

$$[(1-x^2)v']' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] v = 0$$

which is Eq. (*). So apart from a scale factor,

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The scale factor is unity.

Because R's formula for P_n is

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2-1)^n$$

$$P_n^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n n!} \left(\frac{d}{dx}\right)^{n+m} (x^2-1)^n$$

makes perfect sense as long as $n+m \geq 0$.
In fact,

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

Indeed, since m occurs only as m^2, P_n^{-m} would have to be proportional to P_n^m.

$$P_n^0 = P_n$$

Recurrence relations and a generating function are given by Eqs. (12.83-89) of the text.

Parity

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x).$$

$P_n^m(\pm 1) = 0$ for $m \neq 0$ due to the factor $(1-x^2)^{m/2}$.

Orthogonality

$$\int_{-1}^{1} P_p^m(x)\, P_q^m(x)\, dx = \frac{2}{2q+1} \frac{(q+m)!}{(q-m)!} \delta_{p,q}$$

$$\parallel \qquad\qquad \parallel$$

$$\int_0^\pi P_p^m(\cos\theta)\, P_q^m(\cos\theta) \sin\theta\, d\theta$$

Also

$$\int_{-1}^{1} P_n^m(x)\, P_n^k(x) \frac{dx}{(1-x^2)} = \frac{(n+m)!}{m(n-m)!} \delta_{m,k}$$

which is rarely useful.

Spherical Harmonics

$$-\Delta \psi = k^2 f(r)\, \psi \qquad \text{is separable.}$$

The angular part is

$$\frac{\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} + (n+1)n\, \Theta\, \Phi = 0$$

We take

$$\Phi_m(\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}$$

and

$$\Theta_n^m(\theta) = (-1)^m \sqrt{\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}}\; P_n^m(\cos\theta)$$

and set

$$Y_n^m(\theta,\phi) = \Theta_n^m(\theta)\,\Phi_m(\phi)$$

$$= (-1)^m \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}}\; P_n^m(\cos\theta)\, e^{im\phi}$$

so that

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\; Y_n^{m\,*}(\theta,\phi)\, Y_{n'}^{m'}(\theta,\phi) = \delta_{nn'}\,\delta_{mm'}.$$

The Y_n^m's are polynomials in $\sin\theta$ and $\cos\theta$ multiplied by $e^{im\phi}$.

The Y_ℓ^m's are complete for functions defined on the unit sphere. An example is

$$P_\ell(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\,Y_\ell^{m*}(\theta',\phi')$$

More generally, if

$$f(\theta,\phi) = \sum_{mn} a_{mn} Y_n^m(\theta,\phi),$$

or

$$f(\Omega) = \sum_{mn} a_{mn} Y_n^m(\Omega), \quad \text{then}$$

$$\int d\Omega\, f(\Omega)\, Y_n^{m*}(\Omega) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, f(\theta,\phi)\, Y_n^{m*}(\theta,\phi)$$

$$= a_{mn}.$$

So

$$f(\Omega) = \sum_{mn} Y_n^m(\Omega)\int d\Omega'\, f(\Omega')\, Y_n^{m*}(\Omega')$$

$$= \int d\Omega'\, f(\Omega') \sum_{mn} Y_n^m(\Omega)\, Y_n^{m*}(\Omega') \quad \text{so}$$

$$\delta(\Omega-\Omega') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^m(\Omega)\, Y_n^{m*}(\Omega').$$

Another example: for $R > r$,

$$\frac{1}{|\vec{r} - \vec{R}|} = \frac{1}{R} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2m+1} \left(\frac{r}{R}\right)^{l} Y_l^{m*}(\Omega_r) Y_l^{m}(\Omega_R)$$

follows from

$$\frac{1}{|r - R|} = \frac{1}{R} \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^{l} P_l(\hat{r} \cdot \hat{R})$$

and

$$P_n(\hat{r} \cdot \hat{R}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\hat{r}) Y_l^{m*}(\hat{R})$$

$$= \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\hat{R}) Y_l^{m*}(\hat{r}).$$

$$\vec{L} = \frac{\hbar}{i} \vec{r} \times \vec{\nabla}$$

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

$$\vec{L} \cdot \vec{L} Y_l^m(\hat{r}) = L^2 Y_l^m(\hat{r})$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] Y_l^m(\hat{r})$$

$$= l(l+1)\hbar^2 Y_l^m(\hat{r}).$$