1st-order PDEQ's.

\[ \frac{dy}{dx} = f(x, y) = -\frac{P(x, y)}{Q(x, y)} \]

\[ P \, dx + Q \, dy = 0 \]

This system is \underline{exact} if for some \( \phi(x, y) \)

\[ d \phi(x, y) = P(x, y) \, dx + Q(x, y) \, dy \]

In this case,

\[ P(x, y) = \frac{\partial \phi}{\partial x}, \quad Q(x, y) = \frac{\partial \phi}{\partial y}, \]

and so

\[ \frac{\partial P(x, y)}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x} = \frac{\partial Q(x, y)}{\partial x}. \]

When \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0 \), the system is trivially exact, and

\[ 0 = \int_{x_0}^{x} P(x) \, dx + \int_{y_0}^{y} Q(y) \, dy \]

gives the solution \( y(x) \).
With great care, one might find an integrating factor \( \alpha(x,y) \) so that

\[
\alpha(x,y) P(x,y) \, dx + \alpha(x,y) Q(x,y) \, dy = 0
\]

is exact, that is, so that

\[
d\phi(x,y) = \alpha P \, dx + \alpha Q \, dy.
\]

\[
\alpha P = \frac{\partial \phi}{\partial x} \\
\alpha Q = \frac{\partial \phi}{\partial y}
\]

\[
\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial Q}{\partial x}.
\]

Boyle's law arises from the exact

\[
0 = \frac{dV}{V} + \frac{dP}{P}
\]

which we integrate to

\[
0 = \ln \frac{V}{V_0} + \ln \frac{P}{P_0} = \ln \frac{VP}{V_0P_0}
\]

or

\[
VP = V_0 P_0 = k.
\]
But the equation

$$xdg - y dx = 0$$

is not exact.

$$\alpha(x,y) = x^2$$ is an integrating factor.

$$\frac{xdg}{x^2} - \frac{y dx}{x^2} = \frac{dy}{x} - \frac{y}{x^2} dx$$

because now $$P = -y/x^2$$ and $$Q = 1/x$$ so

$$\frac{\partial P}{\partial y} = -\frac{1}{x^2} = \frac{\partial Q}{\partial x},$$ which shows $$\alpha$$ is an integrating factor.

$$0 = \frac{y}{x^2} dx - \frac{dy}{x}$$ is exact. To solve it, we write

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln \frac{y}{y_0} = \ln \frac{x}{x_0} \quad 0 = \ln \frac{y}{y_0} \frac{x}{x_0}$$

so

$$\frac{y}{x} = k \quad y = kx.$$ 

Note that $$\alpha(x,y) = 1/(xy)$$ is another integrating factor for $$xdy - y dx = 0.$$
A function \( f(x, y, z) \) is homogeneous of degree \( n \) if
\[
f(tx, ty, tz) = t^n f(x, y, z),
\]
For instance, \( z^2 \ln(x/y) \) is homogeneous of degree 2 since
\[
(tz)^2 \ln\left(\frac{tx}{ty}\right) = t^2 z^2 \ln\left(\frac{x}{y}\right).
\]
If \( f \) is homogeneous of degree \( n \), then
\[
f(tx, ty, tz) = t^n f(x, y, z)
\]
so
\[
\frac{df(tx, ty, tz)}{dt} = x \frac{df}{dx} + y \frac{df}{dy} + z \frac{df}{dz} = n t^{n-1} f(x, y, z)
\]
Thus \( t = 1 \)
\[
x \frac{df}{dx} + y \frac{df}{dy} + z \frac{df}{dz} = n f(x, y, z),
\]
This is one of Euler's theorems.
Suppose \( P(x, y) \) and \( Q(x, y) \) are homogeneous of degree \( m \) and \( n \). Then in
\[
0 = P(x, y) \, dx + Q(x, y) \, dy,
\]
\text{Let } y = u x, \text{ so } dy = x du \quad \text{and}

0 = P(x, xu) \, dx + Q(x, xu) \, du

0 = x^n \, P(1, u) \, dx + x^{n+1} \, Q(1, u) \, du

0 = x^{n-m-1} \, dx + \frac{Q(1, u)}{P(1, u)} \, du

\text{Now}

\frac{1}{u} \frac{d}{dx} x^{n-m-1} = 0 = \frac{d}{dx} \left( \frac{Q(1, u)}{P(1, u)} \right)

\text{so the PDE is exact in } x \text{ and } u.

0 = x^{n-m} \int_{x_0}^{x} + \int_{u_0}^{u} \frac{Q(1, u')}{P(1, u')} \, du'

\text{gives } u(x).

A PDE that is separable and separated is exact, but an exact PDE is not always separable. Thus

\text{Thus } 0 = P(x) \, dx + Q(y) \, dy \text{ is separable and separated, and } \frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}. \text{ So it's exact.}

But } 0 = x P(x) + x Q(y) \text{ may be exact without being separable.
We may reduce the homogeneous PDE

\[ a(x,y) \frac{\partial \psi}{\partial x} + b(x,y) \frac{\partial \psi}{\partial y} = 0 \]

to an ODE. Let \( \psi = F(\xi) = F(\xi(x,y)) \). Then

\[ \frac{\partial \psi}{\partial x} = \frac{\partial \xi}{\partial x} F' \quad \text{and} \quad \frac{\partial \psi}{\partial y} = \frac{\partial \xi}{\partial y} F' \quad \text{so} \]

\[ 0 = a \frac{\partial \xi}{\partial x} F' + b \frac{\partial \xi}{\partial y} F' \quad \text{is solved for} \quad \psi \]

\[ 0 = a \frac{\partial \xi}{\partial x} + b \frac{\partial \xi}{\partial y} \quad \text{which is a b ad as the original equation. But we may choose} \quad \xi(x,y) \]

to satisfy \( \xi = C \), a constant, so that

\[ d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy = 0 \]

whence

\[ \frac{\partial \xi}{\partial y} = - \frac{\partial \xi}{\partial x} \frac{dx}{dy} \quad \text{and so} \quad 0 \]

\[ 0 = a \frac{\partial \xi}{\partial x} - b \frac{\partial \xi}{\partial y} \frac{dx}{dy} = ady - b dx \]

which is the ODE

\[ \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \]
For the inhomogeneous PDE

$$a \psi_x + b \psi_y + c \psi = 0,$$

where $\psi_x = \psi/\partial x$ and $\psi_y = \psi/\partial y$,
we set $\psi = \Phi(x,y) F(\xi)$ so that

$$F(a \xi_x + b \xi_y + c \xi) + \xi F'(a \xi_x + b \xi_y) = 0$$

Now if we can figure out a solution $\xi$ of the original equation

$$0 = a \xi_x + b \xi_y + c \xi,$$

as well as $\xi = a \xi_x + b \xi_y$ with $\xi = C$
and so

$$\frac{d\xi}{dx} = \frac{b(\xi,y)}{a(\xi,y)}.$$

Then the product $\psi = DF$ is a more general solution of $a \psi_x + b \psi_y + c \psi = 0$. 
Linear 1st-order ODE's.

\[ \frac{dy}{dx} + p(x) y = f(x) = y_x + py, \]

where \( y_x = \frac{dy}{dx}. \)

If \( f(x) = 0, \) then this ODE is homogeneous in \( y. \) The inhomogeneity \( f(x) \) is a source term. The equation \( y_x + py = 0 \)

is linear in \( y \) and \( y_x. \) There are no terms like \( y^2 \) or \( y^3 \) or \( y^n. \)

In this case, there is a general way to find an integrating factor \( \alpha(x). \)

\[ \alpha(x) y_x + \alpha(x) p(x) y = \alpha(x) f(x) \]

such that

\[ \frac{d(\alpha y)}{dx} = \alpha y_x + \alpha p y. \]

\[ \alpha y + \alpha y_x = \alpha p y \rightarrow \]

\[ \frac{\alpha y}{\alpha} = p \]

so we need
So
\[ p(x) = \left[ \ln \alpha(x) \right]_x = \frac{d \ln \alpha}{dx} \]

where
\[ \ln \alpha(x) = \int_{x_0}^{x} p(x') \]

\[ \alpha = e^{\int_{x_0}^{x} p(x') \}

So now
\[ \frac{d}{dx} \left[ e^{g(x)} \right] = \alpha(x) \psi(x) \]

\[ e^{g(x)} \left[ \int_{x_0}^{x} \alpha(x') \psi(x') \right] = \int_{x_0}^{x} \alpha(x') \psi(x') \phi(x') \]

\[ g(x) e^{\int_{x_0}^{x} \alpha(x) \psi(x) \}

\[ g(x) = e^{\int_{x_0}^{x} \alpha(x) \psi(x) \} + C} \]

is the general solution.

Note that
\[ \int_{x_0}^{x} p(x') \]

\[ g_1(x) = C e^{\int_{x_0}^{x} p(x') \}
is the general solution of the homogeneous ODE
\[ 0 = y_x + py \quad \text{or} \quad \frac{y_x}{y} = -p \]

while
\[ y_2(x) = e^{\int px \, dx}, \quad \int px \, dx = \int px \, e^{q x} \]
is a particular solution of the inhomogeneous ODE
\[ 5x + py = f. \]

This is important.

Example. - RL circuit
\[ L \frac{dI}{dt} + RI = V \]

\[ \frac{dI}{dt} + \frac{R}{L} I = \frac{V}{L}, \quad p = \frac{R}{L}, \quad q = \frac{V}{L} \]

\[ \alpha(t) = e^{\int dt' R(t')/L(t')} \]

\[ I(t) = e^{\int dt' R(t')/L(t')} \]

\[ x \left[ \int dt' e^{\frac{V(t')}{L(t')}} + c \right]. \]
If \( R \) is a constant and \( L \) is another constant, then

\[
I(t) = e^{-\frac{tR}{L}} \left[ \int dt' \, e^{\frac{t'R}{L}} \frac{V(t')}{L} + c \right],
\]

If \( V(t) = V_0 \), another constant, then

\[
I(t) = e^{-\frac{tR}{L}} \left[ \frac{V_0}{L} \int dt' \, e^{\frac{t'R}{L}} + c \right]
\]

\[
= e^{-\frac{tR}{L}} \left[ \frac{V_0}{L} \frac{L}{R} e^{\frac{tR}{L}} + c \right]
\]

\[
= e^{-\frac{tR}{L}} \left[ \frac{V_0}{R} + c \right] e^{\frac{tR}{L}}
\]

If \( I(0) = 0 \), then \( c = -\frac{V_0}{R} \) and

\[
I(t) = \frac{V_0}{R} \left( 1 - e^{-\frac{tR}{L}} \right).
\]
The general ODE

\[ \frac{dy}{dx} = f(x, y) \]

may be formally integrated to

\[ y(x) = y(x_0) + \int_{x_0}^{x} f(x', y(x')) \, dx' \]

which invites the Neumann series solution

\[ y_0(x) = y(x_0) \]

\[ y_1(x) = y_0(x_0) + \int_{x_0}^{x} f(x', y_0(x')) \, dx' \]

\[ = y(x_0) + \int_{x_0}^{x} f(x', y(x')) \, dx' \]

\[ y_2(x) = y_1(x_0) + \int_{x_0}^{x} f(x', y_1(x')) \, dx' \]

\[ = y(x_0) + \int_{x_0}^{x} f(x', y(x')) + \int_{x_0}^{x} f(x'', y(x')) \, dx'' \]

This is called Picard's method of successive approximations.
\((s^2 + 1) f' + sf = 0\)

\[
\frac{f'}{f} = -\frac{s}{s^2 + 1}
\]

\[
d \ln f = -\frac{1}{2} \frac{2s}{s^2 + 1} = -\frac{1}{2} d \ln (s^2 + 1)
\]

\[
\frac{\ln f}{f_0} = -\frac{1}{2} \ln \frac{s^2 + 1}{s_0^2 + 1}
\]

\[
\int_0^\infty f'/f_0 = e^{-\frac{1}{2}} \left[ e^{\ln \frac{s^2 + 1}{s_0^2 + 1}} \right]^{\frac{1}{2}}
\]

\[
\frac{f}{f_0} = e^{-\frac{1}{2} \ln \frac{s^2 + 1}{s_0^2 + 1}} = \left( \frac{s^2 + 1}{s_0^2 + 1} \right)^{-\frac{1}{2}} = \sqrt{\frac{s_0^2 + 1}{s^2 + 1}}
\]

\[
f(s) = \frac{c}{\sqrt{s^2 + 1}}
\]

Check:

\[
f' = -\frac{1}{2} c (s^2 + 1)^{-3/2} \sqrt{s} = -\frac{c s}{(s^2 + 1)^{3/2}}
\]

\[
\frac{f'}{f} = -\frac{c s}{(s^2 + 1)^{3/2}} \frac{(s^2 + 1)^{1/2}}{c} = -\frac{s}{s^2 + 1}
\]
Problem 8.2.12  Jumper's equation

\[ \frac{m \, dv}{dt} = mg - bv \]

\[ \frac{dv}{dt} + \frac{b}{m} v = g \quad \text{So} \quad p = \frac{b}{m}, \quad q = g \quad \text{both constants,} \]

\[ v(t) = e^{-\frac{b}{m} t} \left[ \int_{t}^{t'} e^{\frac{b}{m} t'} dt' + c \right] \]

\[ = e^{-\frac{bt}{m}} \left[ mg \frac{e^{\frac{bt}{m}}}{b} + c \right] \]

\[ = mg/b + ce^{-\frac{bt}{m}} \]

If \( v(0) = 0 \), then \( c = -mg/b \) so that

\[ v(t) = mg \left( 1 - e^{-\frac{bt}{m}} \right) \]

The limiting speed is \( \frac{mg}{b} \). Check

\[ mu = \frac{m}{b} \frac{\sqrt{2}}{b} e^{-bt/m} = mg e^{-bt/m} - \frac{mg}{b} \frac{1 - e^{-bt/m}}{b} \]

\[ = mg e^{-bt/m} \]
Review of Separation of Variables

The Helmholtz equation in cartesian coordinates:

$$0 = (\Delta + k^2) \psi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi$$

We let

$$\psi(x, y, z) = X(x) Y(y) Z(z)$$

with

$$X'' = -k_x^2 X, \quad Y'' = -k_y^2 Y, \quad Z'' = -k_z^2 Z$$

and

$$-k^2 = k_x^2 + k_y^2 + k_z^2$$

Example

$$X = \cos(k_x x)$$
$$Y = \sin(k_y y)$$
$$Z = \cos(k_z z)$$

with

$$k_x^2 + k_y^2 + k_z^2 = k^2$$

Pages 106-139 contain my notes on the separation of variables.
Singular Points

Consider the 2nd-order ODE

\[ y'' = f(x, y, y'). \]

Suppose \( y'' = f(x_0, y, y') \) is finite for all finite \( y \) and \( y' \). Then \( x_0 \) is a regular point of the ODE.

But if \( y'' = f(x_0, y, y') \) diverges for any pair of finite values \( (y, y') \), then \( x_0 \) is a singular point of the ODE.

If the ODE is of the form

\[ y'' + P(x) y' + Q(x) y = 0, \]

and both \( P(x_0) \) and \( Q(x_0) \) are finite, then \( x_0 \) is a regular point of the ODE. But if \( P(x_0) \) or \( Q(x_0) \) or both are infinite, then \( x_0 \) is a singular point.

Not all singular points are equally bad.

If \( P(x) \) and \( Q(x) \) diverge as \( x \to x_0 \), but 

\[ (x-x_0) P(x) \] and \( (x-x_0)^2 Q(x) \) remain finite as \( x \to x_0 \), then \( x_0 \) is a regular singular point.
Regular singular points are also called nonessential singular points.

But if either \((x-x_0)P(x)\) or \((x-x_0)^2Q(x)\)
diverges as \(x \to x_0\), then \(x_0\) is an irregular singular point or an essential singularity.

To analyze the point at infinity, we set

\[ z = \frac{1}{x} \quad \text{and look at} \quad z = 0. \]

\[ y' = \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{d^2z}{dx^2} \frac{dy}{dz} = \frac{d^2z}{dx^2} y' \]

\[ = -\frac{1}{x^2} y' = -z^2 y' \]

\[ y'' = \frac{d^2y}{dx^2} = \frac{d^2z}{dx^2} \frac{dy}{dz} = -z^2 \frac{d}{dz} (-z^2 y') \]

\[ = z^4 y'' + 2 z^3 y' \quad \text{so} \]

\[ 0 = y'' + P(z^{-1}) y' + Q(z^{-1}) y \quad \text{becomes} \]

\[ 0 = z^4 y'' + [2z^3 - z^2 P(z^{-1})] y' + Q(z^{-1}) y \quad \text{or} \]

\[ 0 = y'' + \left( \frac{2z - P}{z^2} \right) y' + \frac{Q}{z^2} y. \]
If \( \frac{z^2 - P(z')}{z^2} \) and \( \frac{Q(z')}{z^4} \) remain finite as \( z \to 0 \), then \( x = \infty \) is a regular point of the ODE.

If \( \frac{z^2 - P(z')}{z^2} \) and \( \frac{Q(z')}{z^4} \) remain finite as \( z \to 0 \), then \( x = \infty \) is a regular singular point. Otherwise \( x < \infty \) is an irregular singular point or an essential singularity.

**Problem 8.4.1** Legendre's equation is

\[
(1-x^2) y'' - 2x y' + \ell(\ell+1) y = 0
\]

or

\[
y'' - \frac{2x}{1-x^2} y' + \frac{\ell(\ell+1)}{1-x^2} y = 0
\]

Clearly \( x = \pm 1 \) are singular points. But

\[-(x-1) \frac{2x}{1-x^2} = \frac{2x}{1+x} \to 1 \text{ as } x \to 1\]

and

\[
(x-1)^2 \frac{1}{(1-x^2)} = \frac{x-1}{(x+1)} \to 0 \text{ as } x \to 1
\]

So \( x = 1 \) is regular.
\[- \left( \frac{x + 1}{1 - x^2} \right) \frac{2x}{1 - x} = \frac{2x}{1 - x} \rightarrow -1 \quad \text{as} \quad x \rightarrow -1 \]

\[\left( \frac{x + 1}{1 - x^2} \right) = \frac{x + 1}{1 - x} \rightarrow 0 \quad \text{as} \quad x \rightarrow -1 \]

So \( x_0 = -1 \) is regular.

\[\left( \frac{2z + \frac{2z^{-1}}{1 - z^{-2}}} {1 - z^{-2}} \right) \frac{1}{z^2} = \frac{z}{z} + \frac{z}{z^2 - 1} \]

diverges as \( z \rightarrow 0 \), so \( x_0 = 0 \) is a singular point. But

\[\frac{2z}{z^2 - 1} \rightarrow 0 \quad \text{as} \quad z \rightarrow 0 \]

and

\[\left( \frac{1}{1 - z^{-2}} \right) \frac{1}{z^2} = \frac{1}{z^2 - 1} \rightarrow -1 \quad \text{as} \quad z \rightarrow 0 \]

So \( x_0 = 0 \) is a regular singular point, like \( x_0 = \pm 1 \).
Series Solutions - Frobenius's Method

Consider the general linear, 2nd-order, homogeneous ODE

\[ y'' + P(x)y' + Q(x)y = 0. \]

We try

\[ y(x) = x^r \sum_{k=0}^{\infty} a_k x^k \quad a_0 \neq 0. \]

Since \( P(x) \) and \( Q(x) \) contribute powers of \( x \), this gets messy fast if we try to stay general. So we look at

\[ y'' + w^2 y = 0. \]

\[ y = \sum_{k=0}^{\infty} a_k x^k, \]
\[ y' = \sum_{k=0}^{\infty} (k+1) a_k x^k, \]
\[ y'' = \sum_{k=0}^{\infty} (k+1)(k+2) a_k x^{k+2}. \]

So

\[ \sum_{k=0}^{\infty} (k+1)(k+2) a_k x^{k+2} + w^2 \sum_{k=0}^{\infty} a_k x^k = 0. \]

The most singular term at \( x=0 \) is

\[ a_0 k(k-1) x^k, \]

and so

\[ k(k-1) = 0. \]
This is called the indicial equation.

So \( k = 0 \) or \( k = 1 \).

Let \( j = k - 2 \). Then the ODE is

\[
\sum_{j=-2}^{\infty} a_j (k+j+2)(k+j+1) x^{k+j} + w^2 \sum_{j=0}^{\infty} a_j x^{k+j} = 0
\]

So

\[
a_{j+2} (k+j+2)(k+j+1) + w^2 a_j = 0 \quad or
\]

\[
a_{j+2} = -\frac{w^2}{(k+j+2)(k+j+1)} a_j
\]

which is a two-term recurrence relation.

Case \( k = 0 \): Then the worst term for \( j = -2 \)

\[
a_0 (0)(-1)(-1) x^{-2} = 0
\]

and the next worst term is for \( j = -1 \)

\[
a_1 (0-1+2)(0-1+1) x^{-1} = 0
\]

So \( a_1 \) is arbitrary. We set \( a_1 = 0 \) when

\[
0 = a_3 = a_5 = a_7 \ldots \quad a_{2n+1} = 0
\]
Then for $j = 0$ and $k = 0$, $a_{j+2} = -\frac{\omega^2}{(j+2)(j+1)} a_j$

and so

$$a_2 = -\frac{\omega^2}{2 \cdot 1} \quad a_0 = -\frac{\omega^2}{2} a_0$$

$j = 2$

$$a_4 = -\frac{\omega^2}{4 \cdot 3} \quad a_2 = \frac{\omega^4}{4!} a_0$$

$j = 4$

$$a_6 = -\frac{\omega^2}{6 \cdot 5} a_4 = -\frac{\omega^6}{6!} a_0$$

$$a_{2n} = (-1)^n a_0 \frac{\omega^{2n}}{(2n)!}$$

$$g(x) = a_0 \sum (-1)^n \frac{\omega^{2n}}{(2n)!} = a_0 \cos \omega x.$$  

Case $k = 1$

$$a_{j+2} = -\frac{\omega^2}{(j+3)(j+2)} a_j$$

The worst term is

$$a_0 (1-2+2)(1-2+1)x^{-1} = 0.$$  

The other problematic term

$$a_1 (1-1+2)(1-1+1) x^0 \text{ must vanish.}$$
So we must set $a_1 = 0$ if $k = 1$.

So

$$a_{2n+1} = 0 \quad \text{again},$$

And

$$j = 1 \quad a_2 = -\frac{\omega^2}{3!} a_0 = -\frac{\omega^2}{3!} a_0$$

$$j = 2 \quad a_4 = -\frac{\omega^2}{5!} a_2 = \frac{\omega^2}{5!} a_0$$

$$j = 4 \quad a_6 = -\frac{\omega^2}{7!} a_4 = -\frac{\omega^4}{7!} a_0 \quad \text{etc.}$$

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

So for $k = 1$

$$y(x) = \sum_{n=0}^{8} a_{2n} x^{1+2n}$$

$$= \frac{a_0}{\omega} \sum_{n=0}^{8} (-1)^n \frac{\omega^{2n}}{(2n+1)!} x^{2n+1}$$

$$= \frac{a_0}{\omega} \sum_{n=0}^{8} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!}$$

$$= \frac{a_0}{\omega} \sin \omega x.$$
So we got two independent solutions for the ODE
\[ y'' + wy = 0. \]

Some ODE's require us to work harder for two solutions.

Sometimes one expands about another point
\[ y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k. \]

Suppose we write the ODE as
\[ 0 = L(x) y(x) = y''(x) + P(x) y'(x) + Q(x) y(x), \]
by which we mean
\[ L(x) = \frac{d^2}{dx^2} + P(x) \frac{d}{dx} + Q(x). \]

Then
\[ L(-x) = \frac{d^2}{dx^2} - P(-x) \frac{d}{dx} + Q(-x). \]

If \( L(-x) = \pm L(x) \), then
\[ L(x) y(x) = 0 \] implies \[ L(-x) y(-x) = 0. \]
and \( L(-x) = i L(x) \) further implies

\[
\frac{1}{2} L(x) y(-x) = 0 \quad \text{on}
\]

\[
L(x) y(-x) = 0.
\]

In this case, both \( y(x) \) and \( y(-x) \) is a solution of

\[
L(x) y(x) = 0
\]

and we may resolve \( y(x) \) into even and odd solutions

\[
y(x) = \frac{1}{2} \left[ y(x) + y(-x) \right] + \frac{1}{2} \left[ y(x) - y(-x) \right]
\]

\[
= y_e(x) + y_o(x),
\]

\[
L_e(x) = \frac{d}{dx} \left[ \frac{d}{dx} - \frac{2x}{1-x^2} \frac{d}{dx} + \frac{d^3}{dx^3} \right] = L_e(-x)
\]

\[
L_o(x) = \frac{d}{dx} \left[ \frac{d}{dx} - \frac{x}{1-x^2} \frac{d}{dx} + \frac{m^2}{1-x^2} \right] = L_o(-x)
\]

\[
L_b(x) = \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \frac{x^2 - m^2}{x^2} = L_b(-x)
\]

\[
L_n(x) = \frac{d}{dx} + w^2 = L_n(-x)
\]

These are all even operators.
What can go wrong? Try Bessel's equation

\[ 0 = y'' + \frac{1}{x} y' + \left( \frac{x^2 - n^2}{x^2} \right) y = x^2 y'' + xy' + (x^2 n^2) y \]

Let

\[ y(x) = \sum_{k=0}^{\infty} a_k x^k \]

\[ 0 = \sum_{k=0}^{\infty} a_k \frac{(k+1)(k+2)}{x^k} x^{k+2} + \sum_{k=0}^{\infty} a_2 (k+1) x^{k+3} \]

\[ + \sum_{k=0}^{\infty} a_k x^{k+2} - \sum_{k=0}^{\infty} a_k n^2 x^{k+4} \quad (BE) \]

Set \( k = 0 \) to isolate the terms with \( x^k \):

\[ a_0 \left[ k(k-1) + k - n^2 \right] = 0 \]

So, \( k^2 = n^2 \) is the indicial equation.

The \( x^{k+2} \) terms are

\[ a_k \left[ (k+1)k + k+1 - n^2 \right] = a_k (k+1+n)(k+1-n) \]

Now if \( k = n \), these terms don't vanish unless \( k = m = -1/2 \) or \( k = -n = 1/2 \). In all other cases, we set \( a_1 = 0 \).

Set \( k = m \). The terms with \( x^{m+1} \) in \( (BE) \) are

\[ a_j \left[ (m+j)(m+j-1) + m + j - n^2 \right] + a_{j-2} = 0 \]

\[ a_j \left[ j(j-1) + mj + m(j-1) + m + j \right] + a_{j-2} = 0 \]
\[ a_j \left[ j^2 + 2n_j \right] = -a_{j-2} \]

\[ a_j j(j+2n) = -a_{j-2} \]

\[ a_{j+2} = -\frac{a_j}{(j+2)(j+2+2n)} \]

\[ a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0 m!}{2^2 2! (n+1)!} \]

\[ a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{4 \cdot 2 (2n+4)(2n+2)} \]

\[ 1 \cdot j \cdot 2 \cdot 3 = \frac{a_0 m!}{a_2 \cdot 2 \cdot 2! (n+2)!} \]

\[ a_6 = -\frac{a_4}{6(2n+6)} = -\frac{a_0 m!}{2^6 3! (n+3)!} \]

So,

\[ a_{2p} = (-1)^p \frac{a_0 m!}{2^p p! (m+p)!} \]

\[ \gamma(x) = a_0 \sum_{j=0}^{\infty} \frac{(-1)^j m! x^{n+2j}}{2^j j! (m+j)!} \]
\[ y(x) = a_0 x^n \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (n+j)!} \left( \frac{x}{2} \right)^{n+j} \]

\[ = a_0 x^n \sum_{j=0}^{\infty} J_m(x) \]

Note that \[ J_n(-x) = (-1)^n J_n(x) \]

If \( k = -n \) and \( n \) is not an integer, then we may generate a second solution \( J_n(x) \).

Now:

\[ a_j \left[ (j-n)(j-1-n)-n+j-n^2 \right] + a_{j-2} = 0 \]

\[ a_j \left[ j(j+1)-n(j+1)-n+j \right] + a_{j-2} = 0 \]

\[ a_j \left( j^2-2nj \right) + a_{j-2} = 0 \]

\[ a_j j \left( j-2n \right) + a_{j-2} = 0 \]

\[ a_{j+2} = -\frac{a_j}{(j+2)(j+2-2n)} \]

Now if \( n \geq 0 \) is a positive integer and \( j \) is an even positive integer, there can be trouble when \( j+2 = 2n \).
In this case, one sets

$$J_\omega (x) = (\omega)^\omega \int_0^x y(t) dt$$

and we do not get a second solution.

But what if we expand about a singular point? Consider

$$x^2 y'' = 6y$$

and by

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$\sum a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} = 6 \sum a_\lambda x^{k+\lambda}$$

$$\sum_{\lambda=0}^{\infty} a_\lambda \left[ (k+\lambda)(k+\lambda-1) - 6 \right] x^{k+\lambda} = 0$$

We must have

$$(k+\lambda)(k+\lambda-1) = 6$$

for all $\lambda$. Impossible. Try for $\lambda = 0$

$$k(k-1) = 6 \quad 0 = k^2 - k - 6$$

$$k = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2} = 3 \text{ or } -2.$$ 

So we get two solutions

$$y = x^3 \quad \text{and} \quad y = x^{-2}.$$ 

Here $x = 0$ is a regular singular point of

$$0 = y'' - \frac{6y}{x^2}.$$
Consider the ODE

\[ y'' - \frac{6}{x^3} y = 0. \]

Since \( x^2 (\frac{6}{x^3}) \) diverges as \( x \to 0 \), the point \( x = 0 \) is an essential singularity of this ODE.

Let

\[ y(x) = \sum_{\lambda = 0}^{\infty} a_\lambda x^{\lambda + 1}. \]

\[ x^3 y'' = \sum_{\lambda = 0}^{\infty} (\lambda + 1)(\lambda + 2) a_\lambda x^{\lambda + 3} = \sum_{\lambda = 0}^{\infty} 6 a_\lambda x^{\lambda + 1}. \]

The lowest power of \( x \) is

\[ (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2, \]

so the indicial equation is

\[ \lambda^2 + 3\lambda + 2 = 0. \]

But by construction, \( a_0 \neq 0 \). So we have no solution at all by this series-expansion method.

For the ODE, the point \( x = 0 \)

\[ x^2 y'' + xy' - a^2 y = 0 \]

is a regular singular point.
We try \( y = \sum_{n=0}^{\infty} a_n x^{k+n} \)

\[
\sum x (k+1)(k+2)(k+3) a_k x^{k+1} + (k+1) a_{k+1} x^{k+2} - a_k x^{k+1} = 0
\]

For \( \lambda = 0 \)

\[
k(k-1) + k - a^2 = 0
\]

\[
h^2 = a^2 \quad \text{so} \quad k = \pm a.
\]

But for \( \lambda = 1 \),

\[
[(k+1)k + k+1 - a^2] a_1 = 0
\]

\[
[ k^2 + 2k - a^2 ] a_1 = (2k+1) a_1
\]

\[
= (\pm 1 \pm 2a) a_1 = 0
\]

So \( a_1 = 0 \) unless \( a = \pm 1/2 \).

For \( \lambda = 2 \)

\[
[(k+2)(k+1) + k+2 - a^2] a_2 = 0
\]

\[
= (k^2 + 4k + 4 - a^2) a_2 = 4(k+1) a_2 = 0
\]

So \( a_2 = 0 \) unless \( k = \pm a = -1 \). So apart from special values of \( a \), the only solutions are

\[
y = x^a \quad \text{and} \quad y = x^{-9}.
\]
The ODE

\[ x^2 y'' + y' - a^2 y = 0 \]

has \( x=0 \) as an essential singularity. We try

\[ y = \sum a_k x^{k+2} \]

\[ \sum (k+1)(k+1-1)a_k x^k + (k+2)a_{k+2} x^{k+1} - a^2 a_0 x^k = 0 \]

Now the vanishing of the coefficient of the constant power of \( x \) gives

\[ 0 = k a_0 x^{-1} \quad \text{or} \quad k = 0 \]

\[ \sum [(k+j)(k+j-1)a_j x^{k+j} + (k+j+1)a_{j+1} x^{k+j+1} - a^2 a_j x^{k+j}] = 0 \]

So

\[ (k+j)(k+j-1) - a^2] a_j = -(k+j+1) a_{j+1} \]

\[ a_{j+1} = a_j \frac{a^2 - j(j-1)}{j+1} \]

So if \( a^2 = j(j-1) \), the series terminates, but for general \( a \)

\[ \lim_{j \to \infty} \frac{a_{j+1}}{a_j} = \lim_{j \to \infty} \frac{a}{j} = 0 \]

and the series diverges for all \( x \).
Fuch's Theorem

We always can obtain at least one power-series solution if we expand about a regular point or a regular singular point.

Do we get a second solution too?

1. If the two roots of the indicial equation are equal, we only get one solution.

2. If the two roots differ by a non-integer number, one gets two solutions.

3. If the two roots differ by an integer, the larger of the two roots yields a solution.

If the only set of numbers \( k_1, \ldots, k_n \) for which

\[ 0 = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) \]

for some range of \( x \) is \( k_i = 0 \), for \( i = 1, 2, \ldots, m \), then the functions \( y_1, y_2, \ldots, y_n \) are linearly independent. Otherwise, the \( y_i \) are linearly dependent.

If \( y_1, y_2, \ldots, y_n \) are linearly dependent, then for some \( k_1, k_i, \ldots k_n \), we have
\[ 0 = k_1 y_1(x) + k_2 y_2(x) + \ldots + k_n y_n(x) \]

and so

\[ 0 = k_1 y_1'(x) + k_2 y_2'(x) + \ldots + k_n y_n'(x) \]

and

\[ 0 = k_1 y_1''(x) + k_2 y_2''(x) + \ldots + k_n y_n''(x) \]

\[ \ldots \]

\[ 0 = k_1 y_1^{(n-1)}(x) + k_2 y_2^{(n-1)}(x) + \ldots + k_n y_n^{(n-1)}(x) \]

So if \( Y \) is the matrix

\[ Y_{ij}(x) = y_j^{(i-1)}(x) \]

then \( 0 = Y_{ij}(x) k_j \) or \( Y(x) k = 0 \).

So, the determinant \( |Y(x)| = 0 \) must vanish if the \( y_i \) are linearly dependent.

\[ W(x) = |Y(x)| \]

is called the Wronskian.

A Second Solution

Consider the ODE

\[ y''(x) + P(x) y'(x) + Q(x) y(x) = 0. \]
Suppose that \( y_1(x) \) and \( y_2(x) \) are two linearly independent solutions. Then the Wronskian

\[
W = \begin{vmatrix}
    y_1(x) & y_2(x) \\
    y'_1(x) & y'_2(x)
\end{vmatrix}
\]

\[= y_1 y'_2 - y_2 y'_1 \neq 0.\]

\[
W' = y'_1 y'_2 + y_1 y''_2 - y'_2 y'_1 - y_2 y''_1
\]

\[= y_1 y''_2 - y_2 y''_1\]

must satisfy

\[
W' = y_1 (-Py'_2 - Qy_2) - y_2 (-Py'_1 - Qy_1)
\]

\[
W'(x) = P(x) (y_2 y'_1 - y_1 y'_2) = -P(x) W(x)
\]

So we can integrate the Wronskian

\[
(ln W)' = -P
\]

\[
ln W(x) = - \int_{x}^{\infty} P(x') + C
\]

\[
W(x) = W(a) e^{-\int_{a}^{x} P(x')}
\]

But \( y_2 \frac{d}{dx} \left( \frac{y_2}{y_1} \right) = y_1 y'_2 - y_2 y'_1 = W \)
and so

\[
\frac{d}{dx} \frac{y_2}{y_1} = \frac{W(x)}{y_1^2(x)}
\]

whence

\[
\frac{y_2(x)}{y_1(x)} = \int dx' \frac{W(x')}{y_1^2(x')} + C
\]

or

\[
y_2(x) = y_1(x) \left( \int dx' \frac{W(x')}{y_1^2(x')} + C \right)
\]

\[
= y_1(x) \left( \int dx' \frac{W(x') e^{-\int dx'' P(x'')}}{y_1^2(x')} + C \right)
\]

So given one solution, \( y_1(x) \), one may generate a second solution \( y_2(x) \)

\[
y_2(x) = y_1(x) \int dx' \frac{e^{-\int dx'' P(x'')}}{y_1^2(x')}
\]

apart from additive and multiplicative constants.
An important special case is:

\[ P(x) = 0 \]

so that

\[ y'' + Q(x)y = 0. \]

In this case,

\[ W' = PW = 0 \]

and so the Wronskian \( W \) is a constant

\[ W = y_1y_2' - y_1'y_2 = C \]

In this case, the general formula for \( y_2 \) is just

\[ y_2(x) = y_1(x) \int \frac{1}{y_1(x')} dx' \]

As long as we expand about a regular point or a regular singular point, we always may use the series method to find \( y_1(x) \). Then we may use the Wronskian formula to get a second solution \( y_2(x) \). One may also generate a second solution \( y_2(x) \) by the series method of pages 538-542 of A&W. So a 2nd-order linear homogeneous ODE gives two linearly independent solutions in general. Two, but not three.
Why not freeing?

Suppose $y_i$ for $i = 1, 2, 3$ are three solutions of the 2d-order linear homogeneous ODE

$$0 = y_i'' + Py_i' + Qy_i.$$ 

Then the 3d-order Wronskian $W$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ -Py'_1 - Qy_1 & -Py'_2 - Qy_2 & -Py'_3 - Qy_3 \end{vmatrix} = 0$$

vanishes because the 3d row is a linear combination of the first two rows. So the three solutions must be linearly dependent.
Green's Functions

Example: \[ \nabla \cdot \mathbf{E} = 4\pi \rho \quad \mathbf{E} = -\nabla \phi - \frac{1}{c^2} \frac{\partial \mathbf{A}}{\partial t} \]

In Coulomb gauge, \( \nabla \cdot \mathbf{A} = 0 \), so there

\[ -\Delta \phi = -\nabla^2 \phi = 4\pi \rho. \]

Suppose we have a Green's function such that

\[ -\nabla_i^2 G(x_1, x_2) = \delta(x_1^i - x_2^i) \]

in fact

\[ G(x_1, x_2) = G(\mathbf{x}_1 - \mathbf{x}_2) \quad \Rightarrow \]

\[ -\nabla^2 G(x^3) = \delta(x^3) \]

Then by Green's theorem with \( \phi(x) \) and \( G(x-x') \)

\[ \int (\phi \nabla^2 G - G \nabla^2 \phi) \, d^3 x = \int (\phi \nabla^2 G - G \nabla^2 \phi) \cdot d^3 \sigma \]

And if we assume that \( \phi \) and \( G \) fall off suitably as \( r \to \infty \), and also extend the volume integral over all of space, pushing the surface integral to infinity, then the surface integral vanishes, and we get

\[ -\int \phi(x^3) \nabla^2 G(x-x') \, d^3 x = -\int G(x-x') \nabla^2 \phi(x) \, d^3 x \]

\[ \int \phi(x^3) S(x-x') \, d^3 x \quad \int G(x-x') 4\pi \rho(x) \, d^3 x \]

\[ = \phi(x_1^3) \quad \Phi(x^3) \]
Let $\mathbf{x} \to \mathbf{x}'$ and $\mathbf{x}' \to \mathbf{x}$, we get

$$
\phi(x') = \int G(x', \mathbf{x}) 4\pi \rho(x') \, d^3x',
$$

In fact, since $G$ is defined by

$$
-\nabla^2 G(x_1, x_2) = -\nabla^2 G(x_2, x_1) = \delta^3(x_1, x_2)
$$

$$
= \delta^3(x_2 - x_1) = -\nabla^2 G(x_2 - x_1)
$$

we see that this Green's function is symmetric,

$$
G(x_1, x_2) = G(x_2, x_1).
$$

It is easy to find $G(x')$ such that

$$
-\nabla^2 G(x') = \delta^3(x')
$$

Let

$$
G(x) = \frac{1}{i\mathbf{k} \cdot \mathbf{x} - \mathbf{k}^2}
$$

Then we have

$$
-\nabla^2 G(x') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2} \delta^3(x')
$$

$$
= \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2} = \delta^3(x')
$$

So

$$
g(k) = \frac{1}{k^2}
$$

and

$$
G(x') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2} = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{i\mathbf{k} \cdot \mathbf{x}}{k^2} = \frac{1}{\mathbf{k}^2} \mathbf{x}_x \frac{1}{\mathbf{k}^2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{i\mathbf{k} \cdot \mathbf{x}}{k^2} = \frac{1}{\mathbf{k}^2} \mathbf{x}_x \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{i\mathbf{k} \cdot \mathbf{x}}{k^2}.
$$
\[ G(x^2) = G(r) = \int_0^\infty \frac{dk}{(2\pi)^2} \frac{e^{-ikr}}{ikr} \]

\[ = \frac{1}{r} \int_{-(2\pi)}^{(2\pi)} \frac{dk}{k} e^{\frac{ikr}{2}} \]

\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{z} e^{\frac{iz}{2}} \]

\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{z} e^{\frac{iz}{2}} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{z} e^{\frac{iz}{2}} \]

\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dz}{z} e^{\frac{iz}{2}} = \frac{\pi i}{2\pi i} = \frac{1}{2} \]

\[ G(x^2) = \frac{1}{4\pi r} \]

\[ G(x_1, x_2) = \frac{1}{4\pi |x_1 - x_2|} = G(x_2, x_1). \]

\[ -\nabla_1^2 G(x_1, x_2) = \delta^3(x_1 - x_2) \]

\[ = -\nabla_2^2 G(x_1, x_2), \]
So as we saw weeks ago

\[ \phi(x, t) = \int G(x, x') 4 \pi \rho(x, t) \, d^3 x' \]

\[ = \int \frac{\rho(x', t)}{1 - x' - x''} \, d^3 x' \]

in the Coulomb gauge — and also in electostatics in all gauges.

For the Helmholtz operator

\[ (- \nabla^2 + k^2) G_H(x_1, x_2) = \delta^3(x_1 - x_2) \]

\[ G_H(x_1, x_2) = \frac{\exp(ik|x_1 - x_2|)}{4\pi |x_1 - x_2|} \]

and for

\[ (- \nabla^2 + k^2) G_{\text{MH}}(x_1, x_2) = \delta^3(x_1 - x_2) \]

we get

\[ G_{\text{MH}}(x_1, x_2) = \frac{\exp(-k|x_1 - x_2|)}{4\pi |x_1 - x_2|} \]

both symmetric under \( x_1, x_2 \).
The spherical-harmonic expansion of the electrostatic Green's function is

\[
G(x_1, x_2) = \frac{1}{4\pi |x_1 - x_2|} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{\ell + 1} \frac{\ell}{\ell + 1} \frac{Y^m_l(\theta_1, \phi_1)}{Y^m_l(\theta_2, \phi_2)}
\]

where

\[
t = \begin{cases} 
|x_1| & |x_1| < |x_2| \\
|x_2| & |x_2| < |x_1|
\end{cases}
\]

and

\[
t_2 = \begin{cases} 
|x_2| & |x_2| < |x_1| \\
|x_1| & |x_1| < |x_2|
\end{cases}
\]

This form leads to the multipole expansion of the Coulomb-gauge (or static) electric potential

\[
\phi(x', t) = \int \frac{\rho(y, t) \, d^3 y}{|x' - y|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{\ell + 1} \frac{\ell}{\ell + 1} \frac{Y^m_l(\theta, \phi)}{Y^m_l(\theta', \phi')} \int d^3 y \frac{Y^m_l(\theta', \phi')}{|x' - y|}
\]

in which the charge density \( \rho(y, t) \) is taken to vanish for \( |y| > R \) and in which \( |x' | > 0 \).
One has (12.4a)

\[
\frac{1}{|n_i - n_{i+1}|} = \sum_{l=0}^{s} \frac{v^l}{v > \epsilon} P_e \left( \frac{n_i \cdot n_{i+1}}{n_i \cdot n_{i+1}} \right).
\]

and also

\[
P_e \left( \hat{n}_i \cdot \hat{n}_{i+1} \right) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell} (\theta_i, \phi_i) Y_{\ell}^{\ast} (\theta_{i+1}, \phi_{i+1}).
\]