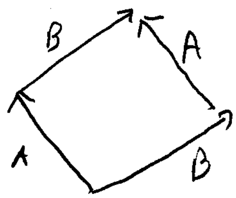
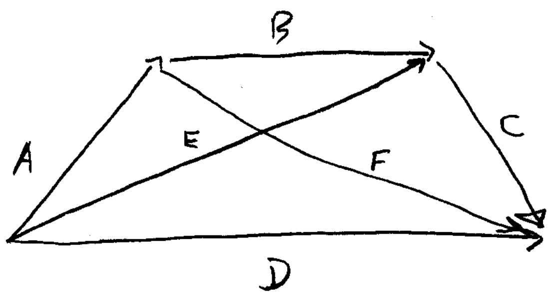
Vectors can be added together

$$A + B = B + A \qquad \text{(abelian)}$$



and multiplied by numbers:

$$C = \alpha A + \beta B.$$



$$A + B = E$$

$$D = E + C$$

$$B + C = F \qquad D = A + F$$

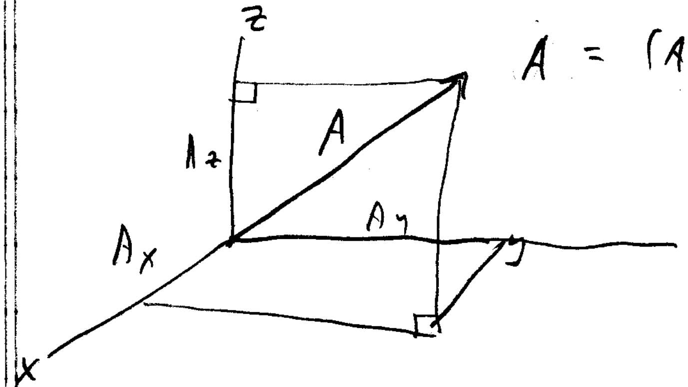
$$(A + B) + C = E + C = D = A + F = A + (B + C)$$

obey distributive law.

$$A - B = A + (-B), \qquad A = E - B$$

Vectors exist independently of any coordinate system.

But if we have one

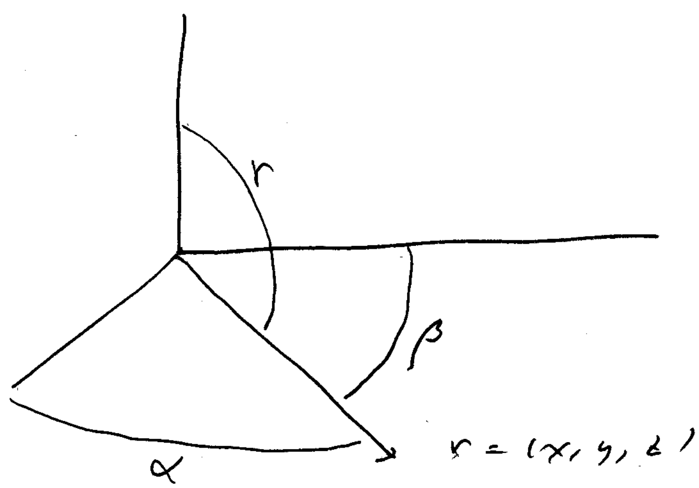


$A = (A_x, A_y, A_z)$

cartesian components
or coordinates

If $\quad r = (x, y, z)$, then

$x = r\cos\alpha \qquad y = r\cos\beta \qquad z = r\cos\gamma$



$$\vec{A} = (A_x, A_y, A_z) = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

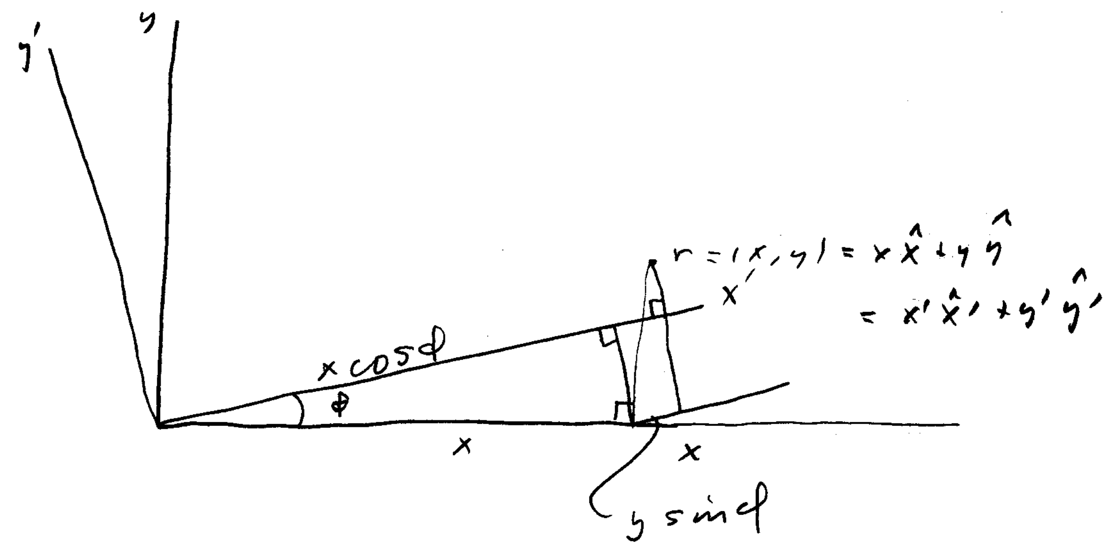
$\hat{x}, \hat{y}, \hat{z}$ are linearly independent,
form basis, are orthogonal

$$\vec{A} \pm \vec{B} = \hat{x}(A_x \pm B_x) + \hat{y}(A_y \pm B_y) + \hat{z}(A_z \pm B_z)$$

Vector $\vec{r}$

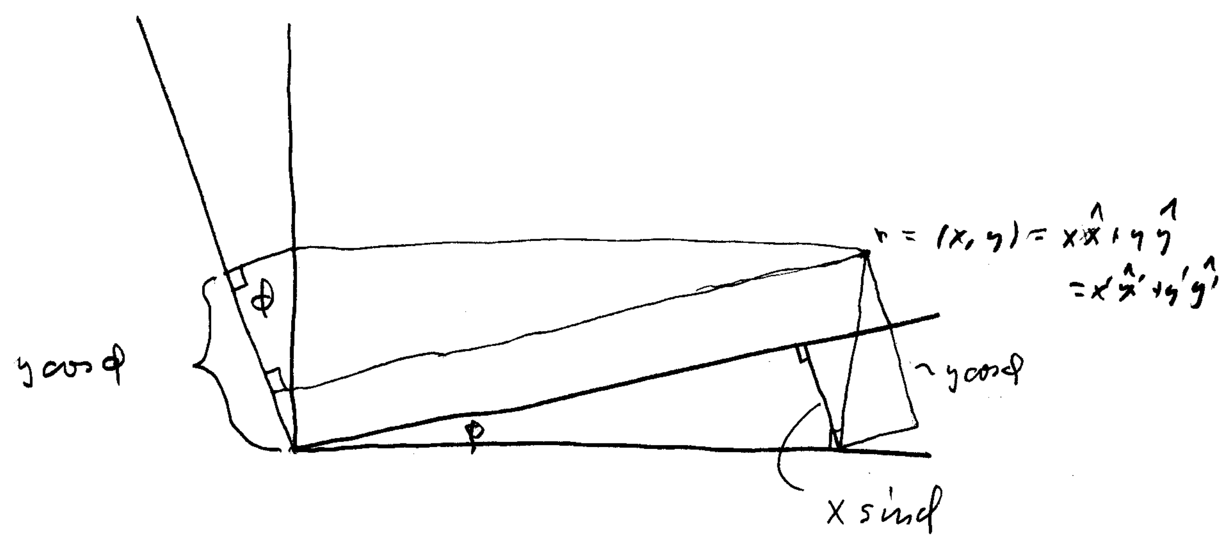
Vector Field $\vec{E}(\vec{r}, t)$

Rotations in 2D



$x' = x\cos\phi + y\sin\phi$

$y' = y\cos\phi - x\sin\phi$



In general, if we rotate the coordinate system by ϕ, and if A is a vector, then

$$A'_x = A_x \cos\phi + A_y \sin\phi$$

$$A'_y = A_y \cos\phi - A_x \sin\phi \,.$$

It is easier to use indices:

$r = (x_1, x_2, x_3)$ etc.

$$r = x_1 \hat{x}_1 + x_2 \hat{x}_2 + x_3 \hat{x}_3 \,.$$

$$V = \sum_{i=1}^{N} V_i \hat{x}_i = \sum_{i=1}^{N} V'_i \hat{x}'_i \,.$$

$$W = \sum W_i \hat{x}_i = \sum W'_i \hat{x}'_i$$

Dot Product of Basis Vectors

ON

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij} = \hat{x}_j \cdot \hat{x}_i \quad \text{symmetric}$$

$$\hat{x}'_i \cdot \hat{x}'_j = \delta_{ij} = \hat{x}'_j \cdot \hat{x}'_i \quad \text{Like } \hat{x} \cdot \hat{y} = 0,\ \hat{z} \cdot \hat{z} = 1.$$

So

$$V \cdot W = \left(\sum_i V_i \hat{x}_i \right) \cdot W$$

$$= \sum_i V_i (\hat{x}_i \cdot W) = \sum_i V_i W_i$$

If the coefficients V_i and W_i are mere numbers, then the dot product is abelian

$$V \cdot W = W \cdot V$$
$$\parallel$$
$$\sum V_i W_i = \sum W_i V_i .$$

But now we see that since

$$V = \sum_{j=1}^{N} V_j' \hat{x}_j' = \sum_{j=1}^{N} V_j \hat{x}_j ,$$ we have

$$\hat{x}_i' \cdot V = \sum_{j=1}^{N} V_j' \hat{x}_i' \cdot \hat{x}_j' = \sum_{j=1}^{N} V_j' \delta_{ij} = V_i'$$

(We assume that both bases $\hat{x}_i$ and $\hat{x}_i'$ are orthonormal.) So

$$V_i' = \hat{x}_i' \cdot V = \hat{x}_i' \cdot \sum V_j \hat{x}_j$$
$$= \sum_{j=1}^{N} V_j (\hat{x}_i' \cdot \hat{x}_j)$$
$$= \sum_{j=1}^{N} \cos(\hat{x}_i', \hat{x}_j) V_j$$
$$= \sum_{j=1}^{N} a_{ij} V_j .$$

Back to 2D:

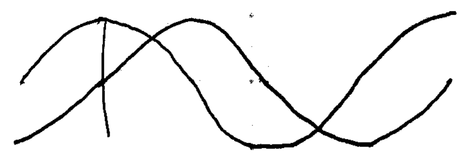
$$x_1' = x_1 \cos\phi + x_2 \sin\phi = \hat{x}_1' \cdot (x_1 \hat{x}_1 + x_2 \hat{x}_2)$$

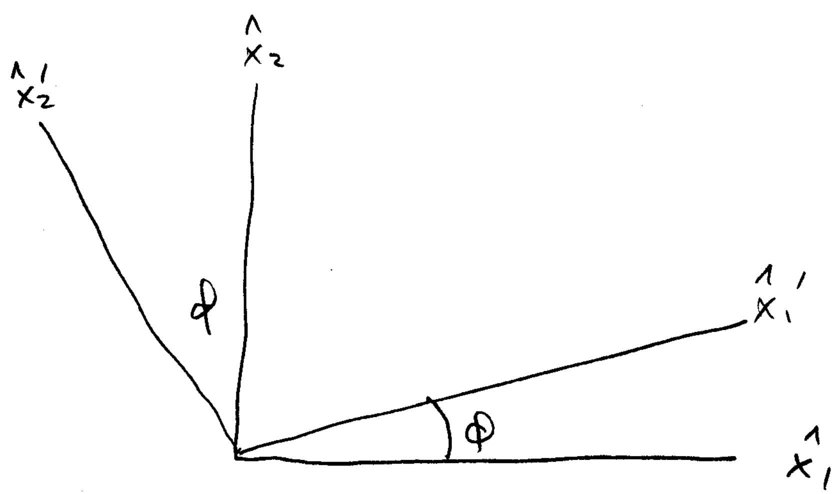
$$x_2' = x_2 \cos\phi - x_1 \sin\phi = \hat{x}_2' \cdot (x_1 \hat{x}_1 + x_2 \hat{x}_2)$$

$$x_1 \cos\phi + x_2 \sin\phi = x_1 \, \hat{x}_1' \cdot \hat{x}_1 + x_2 (\hat{x}_1' \cdot \hat{x}_2).$$

$$\hat{x}_1' \cdot \hat{x}_1 = \cos\phi$$

$$\hat{x}_1' \cdot \hat{x}_2 = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$$





$$\cos\phi = \hat{x}_2' \cdot \hat{x}_2$$

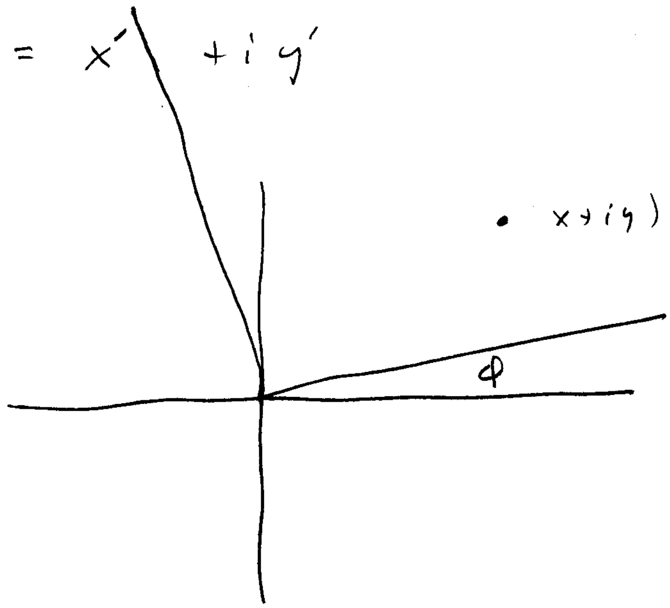
$$-\sin\phi = \hat{x}_2' \cdot \hat{x}_1 = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi.$$

Note how much easier this method is!

$$e^{-i\phi}(x+iy) = (\cos\phi - i\sin\phi)(x+iy)$$

$$= x\cos\phi + y\sin\phi + i(y\cos\phi - x\sin\phi)$$

$$= x' + iy'$$



$$r = \sum x_j \hat{x}_j = \sum x'_j \hat{x}'_j$$

$$x'_i = \hat{x}'_i \cdot r = \sum x'_j \, \hat{x}'_i \cdot \hat{x}'_j = \sum x'_j \delta_{ij}$$

$$= x'_j$$

$$= \hat{x}'_i \cdot r = \hat{x}'_i \cdot \sum_j x_j \hat{x}_j$$

$$= \sum (\hat{x}'_i \cdot \hat{x}_j) x_j \equiv \sum a_{ij} x_j$$

So $a_{ij} = \dfrac{\partial x'_i}{\partial x_j}$

Consider a general vector V in two orthonormal coordinate systems,

$$e_i \cdot e_j = \delta_{ij} = e_i' \cdot e_j',$$

$$V = \sum_{j=1}^{N} V_j e_j = \sum_j V_j' e_j' .$$

Then

$$V_i = e_i \cdot V = \sum_{j=1}^{N} V_j \, e_i \cdot e_j = V_i$$

$$= e_i \cdot V = \sum_{j=1}^{N} V_j' \, e_i \cdot e_j'$$

and also

$$V_j' = e_j' \cdot V = \sum_k V_k \, e_j' \cdot e_k .$$

So actually then for <u>all</u> vectors V,

$$V_i = \sum_{j=1}^{N} V_j' \, e_i \cdot e_j' = \sum_{j=1}^{N} \sum_{k=1}^{N} V_k \, e_j' \cdot e_k \; e_i \cdot e_j'$$

It follows that

$$\sum_{j=1}^{N} e_i \cdot e_j' \; e_j' \cdot e_k = \delta_{ik} \quad \text{so that}$$

$$V_i = \sum_{k=1}^{N} V_k \, \delta_{ik} = V_i .$$

Set $a_{ij} = e_i' \cdot e_j$ so that

$$V_i' = \sum_{j=1}^{N} a_{ij} V_j .$$

Then $e_j \cdot e_k' = e_k' \cdot e_j = a_{kj}$. So

$$V_j = \sum_k V_k' \, e_j \cdot e_k' = \sum_k V_k' a_{kj}$$

And

$$\delta_{ik} = \sum_{j=1}^{N} e_i \cdot e_j' \, e_j' \cdot e_k = \sum_{j=1}^{N} a_{ji} a_{jk}$$

or in matrix notation

$$a^T a = I .$$

In particular

$$x_j' = \sum_i a_{ji} x_i \qquad \text{so} \qquad a_{ji} = \frac{\partial x_j'}{\partial x_i}$$

and

$$x_k = \sum_j a_{jk} x_j' \qquad \text{so} \qquad a_{jk} = \frac{\partial x_k}{\partial x_j'}$$

So

$$\sum_{j=1}^{N} a_{ji} a_{jk} = \sum_{j=1}^{N} \frac{\partial x_j'}{\partial x_i} \frac{\partial x_k}{\partial x_j'} = \frac{\partial x_k}{\partial x_i} = \delta_{ik} = \delta_{ki} .$$

Note also that for any vector V

$$V_i' = e_i' \cdot V = e_i' \cdot \sum V_j e_j$$

$$= \sum e_i' \cdot e_j \, V_j \qquad\qquad V_j = e_j \cdot V$$

$$= \sum e_i' \cdot e_j (e_j \cdot V)$$

$$= \sum_j \sum_k e_i' \cdot e_j \; e_j \cdot e_k' \; V_k'$$

So

$$\delta_{ik} = \sum_j e_i' \cdot e_j \quad e_j \cdot e_k'$$

$$= \sum_j a_{ij} \, a_{kj} \qquad \text{or}$$

$$I = a \, a^T.$$

So $\quad a \, a^T = a^T a = I$. The matrix

$a_{ij} = e_i' \cdot e_j$ is a real orthogonal

matrix.

In what follows, we'll deduce from

$$\delta_{ik} = e_i' \cdot \left(\sum_j e_j \, e_j^T \right) \cdot e_k'$$

$$= e_i \cdot \left(\sum_j e_j' \, e_j^{T\prime} \right) \cdot e_k$$

that

$$\sum_j e_j \, e_j^T = \sum e_j' \, e_j^{T} = I$$

in which the transpose mark means

$$e_j \, e_j^T = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} (0 \ 1 \ 0 \ 0)$$

for example.

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Vector Space

1. $V = W$ means

$$V = \sum e_i V_i = \sum e_i W_i$$

So $e_k \cdot V = V_k = e_k \cdot W = W_k$

$$V_k = W_k,$$

2 $V + W = Q$ means $V_i + W_i = Q_i$

3 $aV = a\sum e_i V_i = \sum (aV_i) e_i$

so $V_i \to aV_i$.

4 $-V = -\sum e_i V_i = \sum (-V_i) e_i$

$$V_i \to -V_i.$$

5 $0 = V = \sum e_i V_i$ means

$$0 = e_k \cdot 0 = \sum e_k \cdot e_i V_i = V_k$$

$$V_k = 0 \qquad k = 1 - N.$$

If the V_i are simply real numbers, then

$$V + W = W + V$$

$$(V + W) + X = V + (W + X)$$

$$a(X + Y) = aX + aY$$

$$a(bX) = (ab)X.$$

0 and $-X$ are unique.

But more generally

$$V = \sum_{j=1}^{\infty} e_i V_i$$

where the e_i are complex and the V_i

can be operators.

HW: 1.1.11, 1.3.4, 1.4.1, 1.4.16, 1.5.5, 1.5.6, 1.5.18

$$V \cdot W = \sum_i e_i V_i \cdot \sum_j e_j W_j$$

$$= \sum_{ij} (e_i \cdot e_j) V_i W_j = \sum \delta_{ij} V_i W_j = \sum_i V_i W_i$$

$$= \left(\sum_i e_i' V_i'\right) \cdot \left(\sum_j e_j' W_j'\right)$$

$$= \sum_{ij} (e_i' \cdot e_j') V_i' W_j' = \sum_{ij} \delta_{ij} V_i' W_j' = \sum_i V_i' W_i'$$

So the dot product is invariant under changes of coordinate systems for real orthogonal transformations.

$$V \cdot V = \left(\sum e_i V_i\right) \cdot \sum e_j V_j$$

$$= \sum_{ij} e_i \cdot e_j \; V_i V_j = \sum \delta_{ij} V_i V_j$$

$$= \sum_i V_i^2 \equiv \|V\|^2 = V^2.$$

as long as the V_i's are real numbers.

Let $C = A + B$. Then

$$C \cdot C = (A+B) \cdot (A+B) = A^2 + B^2 + 2 A \cdot B = C^2$$

So $C^2 = A^2 + B^2 + 2AB \cos\theta$,

in which we wrote

$$A \cdot B = \sum A_i B_i = \sqrt{A^2}\,\sqrt{B^2}\cos\theta$$

where θ is the angle between A and B.

$\perp$

Say $n \cdot r = 0$ for all vectors r in a surface at a point. Then n is a <u>normal</u> vector to the surface at that point.

Cross Product $D=3$ Now

$$e_i \times e_j = \sum_{k=1}^{3} \epsilon_{ijk}\, e_k$$

$$e_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231,\ 312 \\ -1 & ijk = 213,\ 132,\ 321 \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } i=k \end{cases}$$

So

$$e_1 \times e_2 = e_3 \qquad \hat{x} \wedge \hat{y} = \hat{z}$$

$\times \longleftrightarrow \wedge$ when x's are present.

The cross-product is defined as

$$\vec{A} \times \vec{B} = \sum_{ijk} \vec{e}_i \, \epsilon_{ijk} \, A_j \, B_k \, .$$

It is anti-symmetric:

$$B \times A = \sum_{ijk} e_i \, \epsilon_{ijk} \, B_j \, A_k$$

$$= - \sum e_i \, \epsilon_{ikj} \, B_j \, A_k$$

$$= - \sum e_i \, \epsilon_{ikj} \, A_k \, B_j = - A \times B$$

as long as A & B commute.

Now this Levi-Civita tensor obeys

$$\sum_{k=1}^{3} e_{ijk}\, e_{pqk} = \delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}.$$

$$\sum_{pq} \epsilon_{ipq}\, \epsilon_{jpq} = 2\,\delta_{ij}$$

we also have $\sum \delta_{ii} = 3$ $\quad \sum (\epsilon_{ijk})^2 = 6$

$$\sum_{ij} \delta_{ij}\, \epsilon_{ijk} = \sum_{i} \epsilon_{iik} = 0,$$

Let $i = 1,\ j = 2$ then

$$\sum_{k} \epsilon_{ijk}\, \epsilon_{pqk} = \sum \epsilon_{12k}\, \epsilon_{pqk}$$

$$= \epsilon_{pq3} = \begin{cases} 1 & \text{if } pq = 12 \\ -1 & \text{if } pq = 21 \\ 0 & \text{else} \end{cases}$$

So

$$\sum \epsilon_{12k}\, \epsilon_{pqk} = \delta_{1p}\,\delta_{2q} - \delta_{1q}\,\delta_{2p}.$$

$$\vec{dr} = \vec{d\theta} \times (\vec{r} - \vec{p}) \qquad \vec{p} \text{ on axis}$$

$$L = r \times p \qquad !$$

$$F = q(E + v \times B) \qquad \text{mks.}$$

$$v = \omega \times r$$

$$A \cdot (B \times C) = A_i e_i \cdot (B_j e_j) \times (C_k e_k)$$

$$= A_i e_i \cdot (e_j \times e_k) \; B_j C_k$$

$$= A_i e_i \cdot \epsilon_{jkl} e_l \; B_j C_k \qquad \text{cyclic}$$

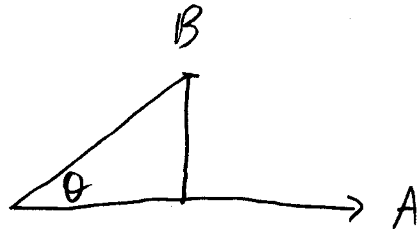
$$= A_i \epsilon_{ijk} B_j C_k$$

$$= \det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

Now

$$\vec{A} \times \vec{B} = |A||B| \sin\theta$$

Choose $\vec{A} = (A, 0, 0)$, $B = (B_x, B_y, 0)$



$$(A \times B)_i = \epsilon_{ijk} A_j B_k$$

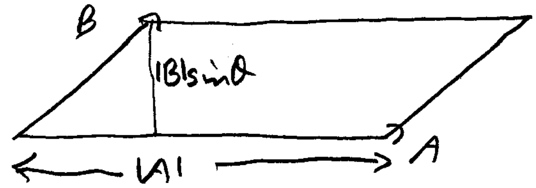
$$= \epsilon_{i1k} A_1 B_k$$

$$= \epsilon_{i12} A_1 B_2$$

$$\vec{A} \times \vec{B} = \vec{e}_3 |A||B| \sin\theta \qquad = \delta_{i3} A_1 B_2$$

$$= \delta_{i3} |A||B| \sin\theta$$

So $|A \times B|$ is area of parallelogram



Direction is that of RH rule.

$$A \times (B + C) = A \times B + A \times C$$

$$(A + B) \times C = A \times C + B \times C$$

$$A \times (y\, B) = y(A \times B) = (y\, A) \times B.$$

the cross-product is bilinear.

Recall

$$\det |M| = \sum \epsilon_{ijk\ldots} \quad M_{1i}\, M_{2j}\, M_{3k} \cdots$$

So

$$\det \begin{vmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \sum \epsilon_{ijk}\, \vec{e}_i\, A_j\, B_k = \vec{A} \times \vec{B}$$

$$(A\times B)\cdot(A\times B) = \sum \epsilon_{ijk}\epsilon_{irs} A_j B_k A_r B_s$$

$$= \sum (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}) A_j B_k A_r B_s$$

$$= A^2 B^2 - (A\cdot B)^2$$

$$= A^2 B^2 - A^2 B^2 \cos^2\theta$$

$$= A^2 B^2 (1-\cos^2\theta) = A^2 B^2 \sin^2\theta$$

$$A\times B = \sum e_i \, \epsilon_{ijk} A_j B_k$$

$$= \sum e_i \, (A\times B)_i \qquad \text{so}$$

$$= \sum e_i' \, (A\times B)_i' \qquad \text{so}$$

$$(A\times B)_i' = e_i' \cdot \sum e_r (A\times B)_r$$

$$= \sum_r e_i'\cdot e_r \, (A\times B)_r \qquad \text{is a vector.}$$

See also 22-23 in text.

Triple Scalar Product

$$A \cdot (B \times C) = \sum A_i \epsilon_{ijk} B_j C_k$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= (B \times C) \cdot A \qquad \text{if coef's commute}$$

$$= - A \cdot (C \times B)$$

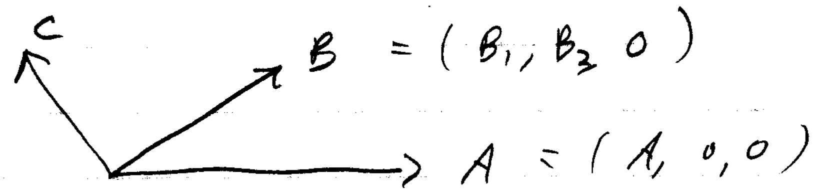
Now ϵ has cyclic symmetry, so

$$A \cdot (B \times C) = \sum \epsilon_{ijk} A_i B_j C_k$$

$$= \sum \epsilon_{jki} B_j C_k A_i = B \cdot (C \times A)$$
$$= - B \cdot (A \times C)$$

$$= \sum \epsilon_{kij} C_k A_i B_j = C \cdot (A \times B)$$
$$= - C \cdot (B \times A).$$

Parallelepiped



Now $\quad A \times B = \vec{e}_3 \, A_1 \, B_2$

So

$$C \cdot (A \times B) = A_1 B_2 C_3$$

$A_1 B_2$ is area of parallelogram in X-Y plane

$A_1 B_2 C_3$ is volume of parallelepiped.

Tni

Triple Vector Product

$$A \times (B \times C) = \sum \vec{e}_i \epsilon_{ijk} A_j (B \times C)_k$$

$$= \sum \vec{e}_i \epsilon_{ijk} A_j \epsilon_{k\ell m} B_\ell C_m$$

$$= \sum \vec{e}_i (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) A_j B_\ell C_m$$

$$= \sum \vec{e}_i B_i A \cdot C - \vec{e}_i C_i A \cdot B$$

$$= \vec{B} (A \cdot C) - \vec{C} (A \cdot B).$$

So you can derive the BAC-CAB rule.

$\vec{\nabla}\phi$ gradient of scalar

Scalar:

$$\phi'(x') = \phi(x)$$

$$\phi'(\vec{x}') = \phi(\vec{x}).$$

The gradient of ϕ is

$$\nabla\phi = \sum \vec{e}_i \frac{\partial \phi(\vec{x})}{\partial x_i}.$$

In another coordinate system

$$(\nabla\phi)' = \sum e_i' (\nabla \phi_i') = \sum_j e_i' \cdot e_j \frac{\partial \phi(x)}{\partial x_i}$$

$$= \sum_j a_{ij} \frac{\partial \phi(x)}{\partial x_j}.$$

which also follows from the chain rule:

$$\frac{\partial \phi'(\vec{x}')}{\partial x_i'} = \frac{\partial \phi(\vec{x})}{\partial x_i'} = \sum_j \frac{\partial x_j}{\partial x_i'} \frac{\partial \phi(\vec{x})}{\partial x_j}$$

Recall $\quad x_j = e_j \cdot x = e_j \cdot \sum e_i' x_i'$

So $\quad \dfrac{\partial x_j}{\partial x_i'} = e_j \cdot e_i' = e_i' \cdot e_j = a_{ij}.$

↓

The gradient of a scalar is a vector.
Works in N dimensions.

$$\vec{F} = -\nabla V$$

$$\vec{\nabla} \frac{q^2}{r} = q^2 \sum \vec{e}_i \frac{\partial \frac{1}{r}}{\partial x_i} = q^2 \sum_i \vec{e}_i \left(-\frac{1}{r^2} \frac{\partial r}{\partial x_i} \right)$$

$$= -q^2 \sum_i e_i \frac{1}{r^2} \frac{\partial \sqrt{\sum x_k^2}}{\partial x_i}$$

$$= -q^2 \sum_i e_i \frac{1}{r^2} \frac{1}{2} \frac{2 x_i}{\sqrt{\sum x_k^2}}$$

$$= -q^2 \sum_i e_i \frac{x_i}{r^3} = -q^2 \frac{\vec{r}}{r^3}$$

Works in N dimensions, but electrodynamics must be derived from

$$- F_{\mu\nu} F^{\mu\nu} + q A_\mu j^\mu + \text{mechanical action}$$

in each dimension.

$$\vec{\nabla} = \sum \vec{e}_i \frac{\partial}{\partial x_i}$$

$$\vec{dx} = \sum \vec{e_i}\, dx_i$$

$$d\phi = \sum_i \frac{\partial \phi}{\partial x_i} dx_i$$

$$= \left(\sum_i \vec{e_i} \frac{\partial \phi}{\partial x_i} \right) \cdot \left(\sum_j e_j\, dx_j \right)$$

$$= \vec{\nabla}\phi \cdot \vec{dx}$$

$\vec{\nabla}\phi$ point in the direction in which ϕ increases most rapidly.

Divergence

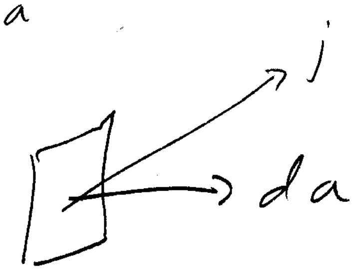
$$\vec{\nabla} \cdot \vec{V} = \left(\sum_i \vec{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_j \vec{e}_j V_j \right)$$

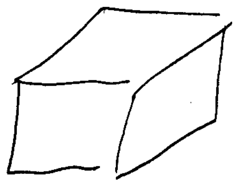
$$= \sum_{ij} e_i \cdot e_j \frac{\partial V_j}{\partial x_i} = \sum \delta_{ij} \frac{\partial V_j}{\partial x_i}$$

$$= \sum_i \frac{\partial V_i}{\partial x_i}$$

as long as the vectors e_i do not themselves depend upon $\vec{x}$.

If $\vec{v}$ is the velocity of a fluid of density ρ, then $\rho \vec{v} = \vec{j}$ is the current. So $j \cdot \vec{da}$ is rate of flow thru $\vec{da}$

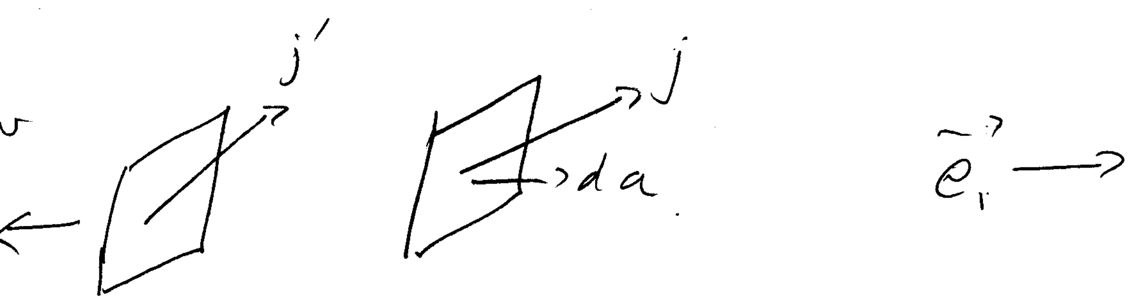




So the sum over the six faces of a cube is the net out flow

$$\text{out flow} = \sum_{i=1}^{6} \vec{j}(x_i) \cdot \vec{da}_i$$

Now



So $$\vec{da}' \cdot j' + \vec{da} \cdot j = da\,(j - j')$$

$$= da \frac{\partial j}{\partial x_1}.$$

So outflow is

$$\text{outflow} = dx_2\, dx_3\, dx_1 \frac{\partial j}{\partial x_1}$$

$$+ dx_3\, dx_1\, dx_2 \frac{\partial j}{\partial x_2} + dx_1\, dx_2\, dx_3 \frac{\partial j}{\partial x_3}$$

outflow $= \vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$

So

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

With $j = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$, we have

$$0 = \frac{\partial j^0}{\partial ct} + \vec{\nabla} \cdot \vec{j} = \frac{\partial j^\mu}{\partial x^\mu} = 0 .$$

Conservation law

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\vec{v} \rho) = 0 ,$$

$$\nabla \cdot (f \vec{V}) = \sum \vec{e}_i \frac{\partial}{\partial x_i} \cdot \left(\sum_j f e_j V_j \right)$$

$$= \sum \delta_{ij} \frac{\partial}{\partial x_j} (f V_i) = \sum_j \frac{\partial f}{\partial x_j} V_j + f \frac{\partial V_j}{\partial x_j}$$

$$= (\vec{\nabla} f) \cdot V + f \cdot \vec{\nabla} \cdot \vec{V} .$$

If $\vec{\nabla} \cdot \vec{B} = 0$, then B is solenoidal.

Curl

$$\nabla \times V = \left(\sum \vec{e}_i \frac{\partial}{\partial x_i}\right) \times \sum \vec{e}_j V_j$$

$$= \sum e_i \times e_j \frac{\partial V_j}{\partial x_i}$$

$$= \sum \vec{e}_k \, \epsilon_{kij} \frac{\partial V_j}{\partial x_i}$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \nabla_1 & \nabla_2 & \nabla_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

The curl is special for $D = 3$.

$$\nabla \times (fV) = \sum \vec{e}_i \frac{\partial}{\partial x_i} \times f \sum e_j V_j$$

$$= \sum_{ijk} \vec{e}_k \, \epsilon_{ijk} \frac{\partial f V_j}{\partial x_i} = f \sum e_k \, \epsilon_{ijk} \frac{\partial V_j}{\partial x_i}$$

$$+ \sum \vec{e}_k \, \epsilon_{ijk} V_j \frac{\partial f}{\partial x_i}$$

$$= f \nabla \times V - \vec{V} \times \nabla f$$

$$= f \nabla \times V + (\nabla f) \times \vec{V}$$

$$\nabla \cdot E = 4\pi\rho$$
for electrostatics.
$$\nabla \times E = 0$$

So we set $E = -\nabla\phi$. Then

$$\nabla \cdot E = \sum \frac{\partial E_i}{\partial x_i} = \sum -\frac{\partial^2\phi}{\partial x_i^2} = 4\pi\rho$$

or

$$\nabla^2\phi = -4\pi\rho$$ Poisson.

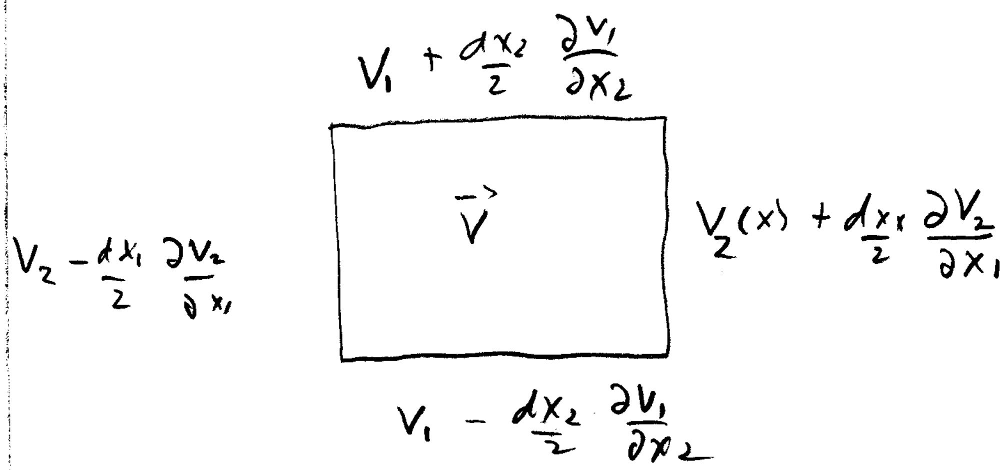
when $\rho = 0$, $\nabla^2\phi = 0$ Laplace

If $E = -\nabla\phi$, then

$$\nabla \times E = \sum \vec{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j} E_k$$

$$= \sum_{ijk} \vec{e}_i \epsilon_{ijk} \frac{\partial}{\partial x_j}\left(-\frac{\partial\phi}{\partial x_k}\right) = 0$$

So the electrostatic field is irrotational.



So
$$\oint \vec{V}\cdot d\vec{x} = dx_1\left(V_1 - \frac{dx_2}{2}\frac{\partial V_1}{\partial x_2}\right)$$

$$+ dx_2\left(V_2 + \frac{dx_1}{2}\frac{\partial V_2}{\partial x_1}\right)$$

$$- dx_1\left(V_1 + \frac{dx_2}{2}\frac{\partial V_1}{\partial x_2}\right)$$

$$- dx_2\left(V_2 - \frac{dx_1}{2}\frac{\partial V_2}{\partial x_1}\right)$$

$$= \frac{dx_1\,dx_2}{2}\left(-\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\right)$$

$$= dx_1 dx_2\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) = dx_1 dx_2 (\nabla \times V)_3$$

So when $\nabla \times V = 0$, one may define

something as
$$G(\vec{x}) = \int_{x_0}^{\vec{x}} d\vec{x}' \cdot \vec{V}(\vec{x}')$$

and be confident that
$$0 = G(\vec{x}) - G(\vec{x}) = \int_{x}^{x} dx' \cdot \vec{V}(x')$$

$$= \int d\sigma \cdot \nabla \times V = 0.$$

A radial fields $F(\vec{r}) = \vec{r} f(\vec{r})$

$$\nabla \times (\vec{r} f(\vec{r})) = f(\vec{r}) \vec{\nabla} \times \vec{r} + (\nabla f) \times \vec{r}$$

But $\vec{\nabla} \times \vec{r} = e_i \epsilon_{ijk} \frac{\partial x_k}{\partial x_j} = 0$ so

$$\vec{\nabla} \times (\vec{r} f(\vec{r})) = \nabla f(\vec{r}) \times \vec{r}.$$

If $f(\vec{r}) = f(|\vec{r}|^2) = f(r^2)$, then

$$\nabla f(r^2) = e_i \frac{\partial f(r^2)}{\partial r^2} \frac{\partial r^2}{\partial x_i} = e_i \frac{\partial f}{\partial r^2} 2x_i$$
$$= 2\vec{r} \frac{\partial f}{\partial r^2} \quad \text{and}$$

So if $F(\vec{r}) = \vec{r}\, f(r^2)$,

$$\nabla \times (F) = \nabla f \times \vec{r} = 2 \frac{\partial f}{\partial r^2} \vec{r} \times \vec{r} = 0.$$

$$\nabla \cdot \nabla \phi = \sum e_i \frac{\partial}{\partial x_i} \cdot \sum e_j \frac{\partial \phi}{\partial x_j}$$

$$= \sum \delta_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = \sum \frac{\partial^2 \phi}{\partial x_i^2}$$

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = \Delta \phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}$$

We saw that $\nabla \times \nabla \phi = 0$.

$$\nabla \cdot (\nabla \times V) = \sum e_i \frac{\partial}{\partial x_i} \sum e_k \frac{\partial V_k}{\partial x_j} \epsilon_{ijk}$$

$$= \sum \delta_{ik} \frac{\partial^2 V_k}{\partial x_i \partial x_j} \epsilon_{ijk} = 0$$

doubly zero.

E & M in vacuum

$\nabla \cdot B = 0$ if $B = \nabla \times A$, $\nabla \cdot \nabla \times A = 0$

$\nabla \cdot E = 0$ $\qquad \rho = 0$

$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t} \to \frac{1}{c}\frac{\partial E}{\partial t}$ if $J = 0$

$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$ $\qquad$ So $\frac{\partial B}{\partial t} = -c \, \nabla \times E$

$$\frac{\partial}{\partial t} \nabla \times B = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \nabla \times \frac{\partial B}{\partial t} = -c \, \nabla \times (\nabla \times E)$$

Now

$$\nabla \times (\nabla \times E) = \sum \vec{e}_i \, \epsilon_{ijk} \frac{\partial}{\partial x_j} \quad \epsilon_{k\ell m} \frac{\partial E_m}{\partial x_\ell}$$

$$= \sum \vec{e}_i \epsilon_{ijk} \epsilon_{k\ell m} \frac{\partial^2 E_m}{\partial x_j \partial x_\ell}$$

$$= \sum \left(\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell} \right) \frac{\partial^2 E_m}{\partial x_j \partial x_\ell} \vec{e}_i$$

$$= \sum \vec{e}_i \left(\frac{\partial^2 E_j}{\partial x_j \partial x_i} - \frac{\partial^2 E_i}{\partial^2 x_j} \right) = \sum_{i,j} -\vec{e}_i \frac{\partial^2 E_i}{\partial x_j}$$

$= -\Delta \vec{E}$ So $\qquad$ since $\nabla \cdot E = 0$

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \Delta \vec{E}$$