Chaos

Henri Poincaré (~1900) studied the three-body problem and found very complicated (chaotic) orbits.

There seem to be four kinds of classical motion:
1) periodic
2) steady (or damped motion that stops)
3) quasi-periodic (mixture of periodic motions, Wi)
4) chaotic

In a system after a transient period.

Examples

\[ x'' + v x' + x^3 - x = g \sin t \quad \text{Exp. B theory} \]

\[ x' \]

\[ x \]

\[ Fe \]

\[ \text{magnets} \]
Data are $t_1, t_2, t_3$

At low flow rate, $\Delta t_m = t_{m+1} - t_m$ is constant, all $\Delta t_m$ are equal.

At a slightly higher rate, the drops come with gaps that alternate $\Delta t_a, \Delta t_b, \Delta t_a, \Delta t_b, \ldots$ so that $\Delta t_{m+1} = \Delta t_m$. This is a period-two sequence.

At still higher flow rates, no regularity is apparent.
Chaotic Rayleigh-Bénard convection occurs when a fluid is placed in a gravitational field between two plates that are kept at constant temperatures with the lower plate hotter by $\Delta T$ above the chaotic threshold. For lower $\Delta T$, the motion is steady convective cellular flow.

\[ \text{Dynamical Systems} \]

\[ \dot{x}_i = F_i(x) \quad \text{and} \quad \dot{x} = \dot{F}(x), \quad x \in \mathbb{R}^n \]

\[ \dot{x}^*(t) = \dot{F}(x^*(t)) \]

The crossings of a suitably oriented plane give rise to a map

\[ \tilde{x}_{n+1} = M(x_n) \]

in some fewer dimensions.
In the system
\[ \dot{\mathbf{x}} = F(\mathbf{x}) \]
chaos can occur only if the dimension \( N \) of the vector \( \mathbf{x} \) exceeds \( N \geq 3 \).

For the invertible map
\[ x_{n+1} = M(x_n) \quad \Rightarrow \quad x_n = M^{-1}(x_{n+1}) \]
chaos occurs only if \( N > 2 \).

If the map is not invertible, then chaos can occur even if \( N = 1 \). An example is
\[ x_{n+1} = r \cdot x_n(1 - x_n) \]
which is not invertible and does exhibit chaos in increasingly striking forms as \( r \) exceeds a number slightly greater than \( r = 3.57 \). By \( r = 4 \), the map is totally chaotic.
Here $x_1 = x_2 = 0$ is an attractor.

Here the circle is an attractor called a limit cycle.

The limit cycle occurs in the van der Pol equation

$$\dot{y} + (\gamma^2 - \eta) y + \omega^2 y = 0$$

which may be written as the first-order system

$$\begin{align*}
\dot{x}_1 &= y \\
\dot{x}_2 &= y
\end{align*}$$

$$\begin{align*}
\dot{x}_1 &= -\omega^2 x_2 - (\gamma^2 - \eta) x_1 \\
\dot{x}_2 &= x_1
\end{align*}$$

The van der Pol equation was introduced in the 1920s to describe a vacuum-tube oscillator circuit.
Fractals

Fractal sets don't have dimensions that are natural numbers. To compute their dimensions one needs a definition of dimension.

The box-counting dimension is as follows: cover the set with line segments, squares, cubes, etc., of edge length \( \varepsilon \). Count how many you need as \( \varepsilon \to 0 \). Call the number of boxes \( N(\varepsilon) \). Then

\[
D_0 = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln (1/\varepsilon)},
\]

Cantor set:

\[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & \frac{1}{3}
\end{array}\]

\[\begin{array}{ccc}
2 & 2 & \\
\end{array}\]

\[E_n = \left(\frac{1}{3}\right)^n \] need \( N(\varepsilon) = 2^n \) boxes.

So

\[D_0 = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \lim_{n \to \infty} \frac{\ln 2}{\ln 3} \approx 0.63.\]

Attractors of fractal dimension are strange.