Note that this trick works for every $(\vec{p}, E)$ that is "on the mass shell"

$$E^2 = c^2 \vec{p}^{\,2} + m^2 c^4 .$$

One needs to use all these $(\vec{p}, E)$'s to get a general solution. Thus the vector potential $\vec{A}(x,t)$ is (here $m=0$)

$$\vec{A}(\vec{x},t) = \sum_{\substack{c^2p^2=E^2 \\ s=1}}^{2} \left(\frac{\hbar^2 c^2}{2VE}\right)^{\frac{1}{2}} \Big[\vec{\epsilon}(p,s)\, e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\, a(p,s) + \vec{\epsilon}^{\,*}(p,s)\, e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar}\, a^{+}(p,s) \Big]$$

in quantum electrodynamics. Take the mean value of $\vec{A}(\vec{x},t)$ in a coherent state $|\alpha\rangle$

$$a(p,s)\,|\alpha\rangle = \alpha(p,s)\,|\alpha\rangle \qquad (\langle\alpha|\alpha\rangle = 1)$$

and get the classical field:

$$\langle\alpha|\,\vec{A}(\vec{x},t)\,|\alpha\rangle = \sum_{\substack{c^2p^2=E^2 \\ s=1}}^{2} \left(\frac{\hbar^2 c^2}{2VE}\right)^{\frac{1}{2}} \Big[\vec{\epsilon}(p,s)\, \alpha(p,s)\, e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar} + \vec{\epsilon}^{\,*}(p,s)\, \alpha^{*}(p,s)\, e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar} \Big].$$

Note that $a(p,s)$ destroys a photon of $\vec{p}, s$, while $a^{+}(p,s)$ makes one. And $\exp(i(p\cdot x - Et)/\hbar)$ is a plane wave. Their product $a(p,s) \exp[i(p x - Et)/\hbar]$ encodes the "wave-particle duality" of quantum mechanics.

Many kinds of homogeneous PDEQ's can be solved by such tricks.

A related trick is the JWKB approximation

$$\psi(x,t) = \sqrt{\rho(x,t)}\; e^{i\frac{S(\vec{x},t)}{\hbar}} .$$

The fast behavior here is $S/\hbar$, not $\rho(\vec{x},t)$. So

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad \text{gives}$$

$$\frac{(\vec{\nabla}S)^2}{2m} + V + \frac{\partial S}{\partial t} = 0$$

which is the Hamilton-Jacobi equation. For a stationary state, one expects

$$S(\vec{x},t) = W(\vec{x}) - Et$$

where $W(\vec{x})$ is Hamilton's characteristic function for which

$$\frac{(\nabla W)^2}{2m} + V = E .$$

Let us switch to natural units $\hbar = c = 1$ to discuss non-linear PDEQ's.

$$\Box \Phi = V'(\Phi)$$

Suppose $\Phi(x,t) = \phi(p \cdot x - E t)$, then

$$(p^2 - E^2)\,\phi'' = V'(\phi) \qquad \text{and so}$$

$$(p^2 - E^2)\,\phi''\phi' = V'(\phi)\,\phi'$$

$$(p^2 - E^2)\,\frac{\phi'^2}{2} = V(\phi) - \mathcal{E}$$

where $\mathcal{E}$ is a constant of integration

$$\mathcal{E} = m^2 \frac{\phi'^2}{2} + V(\phi)$$

This is like ordinary classical mechanics.

$$\phi'^2 = \frac{2}{m^2}\left(\mathcal{E} - V(\phi)\right)$$

Let $z = p \cdot x - E t$, then

$$\frac{d\phi}{dz} = \frac{\sqrt{2}}{m}\sqrt{\mathcal{E} - V(\phi)} \qquad \text{so}$$

$$\int \frac{d\phi}{\sqrt{\mathcal{E} - V(\phi)}} = \int \frac{\sqrt{2}}{m}\, dz = \frac{\sqrt{2}}{m}(z - z_0)$$

where z_0 is another integration constant.

The simplest case is the time-independent case for which $E = 0$. Then if $p = (1,0,0)$ & $\varepsilon = 0$,

$$\frac{\phi'^2}{2} = V(\phi).$$

As an example, let's take

$$V(\phi) = \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2.$$

Then

$$\frac{\phi'^2}{2} = \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2 \qquad \text{or}$$

$$\phi' = \pm\lambda\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right) \qquad \text{whence}$$

$$\int \frac{d\phi}{\phi^2 - \frac{\mu^2}{\lambda^2}} = -\frac{\lambda}{\mu}\tanh^{-1}\left(\frac{\phi\lambda}{\mu}\right) = \pm\lambda(x - x_0)$$

or

$$\tanh^{-1}\left(\phi\frac{\lambda}{\mu}\right) = \mp\mu(x_0 - x) \qquad \text{or}$$

$$\phi\frac{\lambda}{\mu} = \mp\tanh\mu(x_0 - x) \qquad \text{or}$$

$$\phi(x) = \pm\frac{\mu}{\lambda}\tanh\mu(x - x_0)$$

K. C. Phys. Lett. 53B (1974) 174.

This is a soliton in the loose sense — there is some scattering.

The energy of this field theory is

$$H = \int d^3x \; \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2$$

so the solution

$$\phi(x) = \frac{\mu}{\lambda} \tanh \mu(x - x_0)$$

is localized near $x = x_0$.

1st-order PDEQ's.

$$\frac{dy}{dx} = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$$

$$P\,dx + Q\,dy = 0$$

This system is <u>exact</u> if for some $\varphi(x,y)$

$$d\varphi(x,y) = P(x,y)\,dx + Q(x,y)\,dy.$$

In this case,

$$P(x,y) = \frac{\partial \varphi}{\partial x} \qquad Q(x,y) = \frac{\partial \varphi}{\partial y},$$

and so

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial^2 \varphi}{\partial x y} = \frac{\partial Q(x,y)}{\partial x}.$$

When $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$, the system is trivially exact, and

$$0 = \int_{x_0}^{x} P(x)\,dx + \int_{y_0}^{y} Q(y)\,dy$$

gives the solution $y(x)$.

With great luck, one might find an integrating factor $\alpha(x,y)$ so that

$$\alpha(x,y)\, P(x,y)\, dx + \alpha(x,y)\, Q(x,y)\, dy = 0$$

is exact, that is, so that

$$d\phi(x,y) = \alpha P\, dx + \alpha Q\, dy.$$

$$\alpha P = \frac{\partial \phi}{\partial x} \qquad \alpha Q = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial \alpha P}{\partial y} = \frac{\partial^2 \phi}{\partial xy} = \frac{\partial \alpha Q}{\partial x}.$$

Boyle's law arises from the exact DEQ

$$0 = \frac{dV}{V} + \frac{dP}{P}$$

which we integrate to

$$0 = \ln\frac{V}{V_0} + \ln\frac{P}{P_0} = \ln\frac{VP}{V_0 P_0}$$

or $\quad VP = V_0 P_0 = k.$

But the equation

$$x\,dy - y\,dx = 0$$ is not exact.

$\alpha(x,y) = x^{-2}$ is an integrating factor

$$\frac{x\,dy}{x^2} - \frac{y\,dx}{x^2} = \frac{dy}{x} - \frac{y}{x^2}\,dx$$

because now $P = -y/x^2$ and $Q = 1/x$ so

$$\frac{\partial P}{\partial y} = -\frac{1}{x^2} = \frac{\partial Q}{\partial x},$$ which shows that

$$0 = \frac{y}{x^2}\,dx - \frac{dy}{x}$$ is exact. To solve it,

we write

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln\frac{y}{y_0} = \ln\frac{x}{x_0} \qquad 0 = \ln\frac{y}{x}\frac{x_0}{y_0}$$

So

$$\frac{y}{x} = k \quad \text{or} \quad y = kx.$$

Note that $\alpha(x,y) = 1/(xy)$ is another integrating factor for $x\,dy - y\,dx = 0$.

A function $f(x,y,z)$ is <u>homogeneous</u> of degree n if

$$f(tx,ty,tz) = t^n f(x,y,z).$$

For instance, $z^2 \ln(x/y)$ is homogeneous of degree 2 since

$$(tz)^2 \ln\left(\frac{tx}{ty}\right) = t^2 z^2 \ln\left(\frac{x}{y}\right).$$

If f is homogeneous of degree n, then

$$f(tx,ty,tz) = t^n f(x,y,z)$$

so

$$\frac{d f(tx,ty,tz)}{dt} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n t^{n-1} f(x,y,z)$$

or at $t=1$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n f(x,y,z).$$

This is one of Euler's theorems.

Suppose $P(x,y)$ and $Q(x,y)$ are homogeneous of degree n and m. Then in

$$0 = P(x,y)\,dx + Q(x,y)\,dy,$$

let $y = v x$, so $dy = x\,dv$ and

$$0 = P(x, xv)\,dx + Q(x, xv)\,x\,dv$$

$$0 = x^n P(1,v)\,dx + x^{m+1} Q(1,v)\,dv$$

$$0 = x^{n-m-1}\,dx + \frac{Q(1,v)}{P(1,v)}\,dv$$

Now

$$\frac{\partial x^{n-m-1}}{\partial v} = 0 = \frac{\partial Q(1,v)/P(1,v)}{\partial x}$$

so the DEQ is exact in x and v.

$$0 = \left.\frac{x^{n-m}}{n-m}\right]_{x_0}^{x} + \int_{v_0}^{v} dv' \, \frac{Q(1,v')}{P(1,v')}$$

gives $v(x)$.

A DEQ that is separable and separated is exact, but an exact DEQ is not always separable. Thus

$0 = P(x)\,dx + Q(y)\,dy$ is separable and separated, and $\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$. So it's exact.

But $0 = \alpha P\,dx + \alpha Q\,dy$ may be exact without being separable.

We may reduce the homogeneous PDE

$$a(x,y)\,\frac{\partial\psi}{\partial x} + b(x,y)\,\frac{\partial\psi}{\partial y} = 0$$

to an ODE. Let $\psi = F(\xi) = F(\xi(x,y))$.
Then

$$\frac{\partial\psi}{\partial x} = \frac{\partial\xi}{\partial x}\,F' \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \frac{\partial\xi}{\partial y}\,F', \quad \text{so}$$

$$0 = a\,\frac{\partial\xi}{\partial x}\,F' + b\,\frac{\partial\xi}{\partial y}\,F' \quad \text{is solved if}$$

$$0 = a\,\frac{\partial\xi}{\partial x} + b\,\frac{\partial\xi}{\partial y}$$ which is as bad as the

original equation. But we may choose $\xi(x,y)$ to satisfy $\xi = c$, a constant, so that

$$d\xi = \frac{\partial\xi}{\partial x}\,dx + \frac{\partial\xi}{\partial y}\,dy = 0$$

whence

$$\frac{\partial\xi}{\partial y} = -\,\frac{\partial\xi}{\partial x}\,\frac{dx}{dy} \quad \text{and so}$$

$$0 = a\,\frac{\partial\xi}{\partial x} - b\,\frac{\partial\xi}{\partial x}\,\frac{dx}{dy} \quad \text{or} \quad a\,dy = b\,dx$$

which is the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

For the inhomogeneous PDE

$$a\psi_x + b\psi_y + c\psi = 0,$$

where $\psi_x = \partial\psi/\partial x$ and $\psi_y = \partial\psi/\partial y$,
we set $\psi = \mathcal{I}(x,y)\,F(\xi)$ so that

$$F\left(a\mathcal{I}_x + b\mathcal{I}_y + c\mathcal{I}\right) + \mathcal{I}F'\left(a\xi_x + b\xi_y\right) = 0$$

Now if we can figure out a solution $\mathcal{I}$ of the original equation

$$0 = a\mathcal{I}_x + b\mathcal{I}_y + c\mathcal{I},$$

as well as of $0 = a\xi_x + b\xi_y$ with $\xi = C$
and so

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

then the product $\psi = \mathcal{I}F$ is a more general solution of $a\psi_x + b\psi_y + c\psi = 0$.

Linear 1st-order ODE's.

$$\frac{dy}{dx} + p(x)\,y = q(x) = y_x + py ,$$

where $y_x = dy/dx$.

If $q(x) = 0$, then this ODE is homogeneous in y. The inhomogeneity $q(x)$ is a source term. The equation

$$y_x + py = q$$

is linear in y and y_x. There are no terms like y^2 or y^3 or y_x^4.

In this case, there is a general way to find an integrating factor $\alpha(x)$.

$$\alpha(x)\,y_x + \alpha(x)\,p(x)\,y = \alpha(x)\,q(x)$$

such that

$$\frac{d}{dx}(\alpha y) = \alpha y_x + \alpha p y .$$

$$\parallel$$

$$\alpha_x y + \alpha y_x$$ so we need

$$\alpha_x y = \alpha p y$$ or

$$\frac{\alpha_x}{\alpha} = p$$

So

$$p(x) = \left[\ln \alpha(x)\right]_x = \frac{d \ln \alpha}{dx},$$

whence

$$\ln \alpha(x) = \int^x dx' \; p(x')$$

or

$$\alpha = e^{\int^x dx' \; p(x')}$$

So now

$$\frac{d}{dx}\left[e^{\int^x dx' \; p(x')} y(x) \right] = \alpha(x) \, g(x)$$

or

$$\left. e^{\int^x dx' \; p(x')} y(x) \right]_{x_1}^{x} = \int^x dx' \; \alpha(x') \, g(x').$$

$$y(x) \, e^{\int^x dx' \; p(x')} - C = \int^x dx' \; e^{\int^{x'} dx'' \; p(x'')} g(x')$$

or

$$y(x) = e^{-\int^x dx' \; p(x')} \left[\int^x dx' \, e^{\int^{x'} dx'' \; p(x'')} g(x') + C \right]$$

is the general solution.

Note that

$$y_1(x) = C \; e^{-\int^x dx' \; p(x')}$$

is the general solution of the homogeneous ODE

$$0 = y_x + p y \qquad \text{or} \qquad \frac{y_x}{y} = -p$$

while

$$y_2(x) = e^{-\int^x dx'\, p(x')} \int^x dx'\, e^{\int^{x'} dx''\, p(x'')} q(x'')$$

is a particular solution of the inhomogeneous ODE

$$y_x + p y = q.$$

This is important.

Example. RL circuit

$$L\, I_t + R I = V$$

$$I_t + \frac{R}{L} I = \frac{V}{L} \qquad p = R/L$$
$$q = V/L$$

$$\alpha(t) = e^{\int^t dt'\, R(t')/L(t')}$$

$$I(t) = e^{-\int^t dt'\, R(t')/L(t')} \times \left[\int^t dt'\, e^{\int^{t'} dt''\, R(t'')/L(t'')} \frac{V(t')}{L(t')} + c \right].$$

If $R(t') = R$ a constant and $L(t') = L$ another constant, then

$$I(t) = e^{-tR/L}\left[\int^t dt'\, e^{t'R/L}\,\frac{V(t')}{L} + c\right],$$

If $V(t) = V_0$, another constant, then

$$I(t) = e^{-tR/L}\left[\frac{V_0}{L}\int^t dt'\, e^{t'R/L} + c\right]$$

$$= e^{-tR/L}\left[\frac{V_0}{L}\,\frac{L}{R}\,e^{tR/L} + c\right]$$

$$= \frac{V_0}{R} + c\, e^{-tR/L}$$

If $I(0) = 0$, then $c = -\frac{V_0}{R}$ and

$$I(t) = \frac{V_0}{R}\left(1 - e^{-tR/L}\right).$$

The general ODE

$$\frac{dy}{dx} = f(x,y)$$ may be formally integrated to

$$y(x) = y(x_0) + \int_{x_0}^{x} dx'\, f(x', y(x'))$$

which invites the Neumann series solution

$$y_0(x) = y(x_0)$$

$$y_1(x) = y_0(x_0) + \int_{x_0}^{x} dx'\, f(x', y_0(x'))$$

$$= y(x_0) + \int_{x_0}^{x} dx'\, f(x', y(x_0))$$

$$y_2(x) = y_1(x_0) + \int_{x_0}^{x} dx'\, f(x', y_1(x'))$$

$$= y(x_0) + \int_{x_0}^{x} dx'\, f\left(x', y(x_0) + \int_{x_0}^{x'} dx''\, f(x'', y(x_0))\right)$$

This is called Picard's method of successive approximations.

Prob. 8.2.2

$$(s^2+1)f' + sf = 0$$

$$\frac{f'}{f} = -\frac{s}{s^2+1}$$

$$d\ln f = -\frac{1}{2}\frac{2s}{s^2+1} = -\frac{1}{2}\,d\ln(s^2+1)$$

$$\ln\frac{f}{f_0} = -\frac{1}{2}\ln\frac{s^2+1}{s_0^2+1}$$

$$f/f_0 = e^{-\frac{1}{2}\ln\frac{s^2+1}{s_0^2+1}} = \left[e^{\ln\frac{s^2+1}{s_0^2+1}}\right]^{-\frac{1}{2}}$$

$$= \left(\frac{s^2+1}{s_0^2+1}\right)^{-\frac{1}{2}} = \sqrt{\frac{s_0^2+1}{s^2+1}}$$

So

$$f(s) = \frac{c}{\sqrt{s^2+1}},$$

Check:

$$f' = -\tfrac{1}{2}c\,(s^2+1)^{-3/2}\,2s = -\frac{cs}{(s^2+1)^{3/2}}$$

$$\frac{f'}{f} = -\frac{cs}{(s^2+1)^{3/2}}\,\frac{(s^2+1)^{1/2}}{c} = -\frac{s}{s^2+1}.$$

Problem 8.2.12 Jumper's equation

$$m \frac{dv}{dt} = mg - bv$$

$$\frac{dv}{dt} + \frac{b}{m} v = g$$ So $p = \frac{b}{m}$, $q = g$ both constants.

$$v(t) = e^{-\frac{b}{m}t} \left[\int^t dt' \, e^{t'b/m} g + c \right]$$

$$= e^{-bt/m} \left[mg \frac{e^{bt/m}}{b} + c \right]$$

$$= mg/b + c\, e^{-bt/m}$$

If $v(0)=0$, then $c = -mg/b$ so that

$$v(t) = \frac{mg}{b} \left(1 - e^{-bt/m} \right).$$

The limiting speed is $\frac{mg}{b}$. Check.

$$m v_t = + \frac{m g^2}{b} \frac{b}{m} e^{-bt/m} = mg e^{-bt/m} = mg - b \frac{mg}{b} (1 - e^{-bt/m})$$

$$= mg e^{-bt/m}.$$