In Section 6.5 of A&W, we saw that a function $f(z)$ analytic in an annulus with center at $z_0$ possesses the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n \quad \text{(6.69)}$$

where

$$C_n = \frac{1}{2\pi i} \oint_{C} \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

The contour $C$ lies in the annulus.

Setting $z_0 = 0$ and choosing the annulus to enclose the unit circle, we have

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} C_n e^{in\theta}$$

$$C_n = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(e^{i\theta'}) e^{in\theta'}}{e^{i(n+1)\theta'}} d\theta'$$

On

$$C_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-in\theta} f(e^{i\theta}) d\theta.$$ 

By construction, $f(e^{i\theta})$ is periodic in $\theta$

$$f(e^{i(\theta+2\pi n)}) = f(e^{i\theta}) \text{ with period } 2\pi.$$
If we use the notation \( f(\theta) \) for \( f(e^{i\theta}) \), then we may write \( f(\theta) \) as the Fourier series

\[
f(\theta) = \sum_{n=-\infty}^{\infty} C_n e^{i n \theta}
\]

with Fourier coefficients \( C_n \),

\[
C_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i n \theta} f(\theta) \, d\theta,
\]

as long as we understand that \( f(\theta) \) is periodic with period \( 2\pi \),

\[
f(\theta + 2\pi n) = f(\theta).
\]

Combining the equations for \( f(\theta) \) and \( C_n \), we get

\[
f(\theta) = \sum_{n=-\infty}^{\infty} e^{i n \theta} \int_{0}^{2\pi} e^{-i n \theta'} f(\theta') \, d\theta'
\]

\[
= \int_{0}^{2\pi} f(\theta') \sum_{n=-\infty}^{\infty} \frac{e^{i n (\theta - \theta')}}{2\pi}
\]

from which we may infer the completeness relation

\[
\delta(\theta - \theta') = \sum_{n=-\infty}^{\infty} \frac{e^{i n (\theta - \theta')}}{2\pi}
\]

for the functions \( \exp(i n \theta) / \sqrt{2\pi} \).
This derivation shows that Fourier-series expansion & the completeness of the functions

\[ \langle \theta \rangle = \frac{e^{i\theta}}{\sqrt{2\pi}} = F_n(\theta) \]

hold for functions \( f(\theta) \) that are related to functions analytic on the unit circle.

Actually, they hold for a broader class of functions, including those that are piecewise continuous or square integrable, but convergence is "in the mean"—more later.

The exponential form of Fourier series is simple and complex. The more common form is complex and real. To find it, we express \( f(\theta) \) and \( X_n \) in terms of their real and imaginary parts:

\[ f(\theta) = u(\theta) + i v(\theta) \]

\[ C_n = x_n + i y_n \text{, so that} \]

\[ u(\theta) + iv(\theta) = \sum_{n=-\infty}^{\infty} (x_n + i y_n)(\cos n\theta + i \sin n\theta) \]

\[ = x_0 + y_0 + \sum_{n=1}^{\infty} (x_n + x_{-n}) \cos n\theta - (y_n - y_{-n}) \sin n\theta \]

\[ + i \sum_{n=1}^{\infty} (y_n + y_{-n}) \cos n\theta + (x_n - x_{-n}) \sin n\theta \]
So \( u(\theta) \) has the expansion

\[
u(\theta) = x_0 + \sum_{n=1}^{\infty} \left( (x_n + x_{-n}) \cos n \theta - (y_n + y_{-n}) \sin n \theta \right)\]

and \( v(\theta) \) has

\[
v(\theta) = y_0 + \sum_{n=1}^{\infty} \left( (x_n + y_{-n}) \cos n \theta + (x_{-n} - y_n) \sin n \theta \right),
\]

where

\[
c_n = x_n + i y_n = \int_0^{2\pi} \left( \cos n \theta - i \sin n \theta \right) (u(\theta) + iv(\theta)) \frac{d\theta}{2\pi}
\]

or

\[
x_n = \int_0^{2\pi} \cos n \theta \, u(\theta) + \sin n \theta \, v(\theta) \frac{d\theta}{2\pi}
\]

and

\[
y_n = \int_0^{2\pi} \cos n \theta \, u(\theta) - \sin n \theta \, v(\theta) \frac{d\theta}{2\pi}
\]

The book's notation is

\[
a_n = x_n + x_{-n} = \int_0^{2\pi} \cos n \theta \, u(\theta) \tag{14.11}
\]

\[
b_n = -(y_n - y_{-n}) = \int_0^{2\pi} \sin n \theta \, u(\theta), \tag{14.12}
\]

whence

\[
u(\theta) = \frac{x_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \theta + \sum_{n=1}^{\infty} b_n \sin n \theta. \tag{14.1}
\]
If now we set
\[ \chi_n = y_{n+1} + y_{n-1} = \int_0^{2\pi} \frac{d\theta}{\pi} \cos n\theta \, u(\theta) \]
and
\[ \beta_n = x_n - x_{n-1} = \int_0^{2\pi} \frac{d\theta}{\pi} \sin n\theta \, u(\theta) \]
then the Fourier series for \( u(\theta) \) is
\[ u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \beta_n \sin n\theta. \]

We have from (14.1) & (14.11-12)
\[ u(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \, u(\theta') \]
\[ + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} d\theta' \left[ \cos(n\theta) \cos(n\theta') + \sin(n\theta) \sin(n\theta') \right] u(\theta') \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \, u(\theta') + \int_0^{2\pi} d\theta' \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n(\theta - \theta')) \, u(\theta') \]
whence
\[ \delta(\theta - \theta') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos[n(\theta - \theta')]. \]
\[ \delta(\theta - \theta') = \sum_{n = -\infty}^{\infty} \cos m(\theta - \theta') \frac{\cos nx}{2 \pi} \]

All these functions, \( e^{im\theta} \) and its real and imaginary parts are orthogonal and normalizable on the interval \([0, 2\pi]\).

\[ \delta_{mn} = \frac{2\pi}{\int_{0}^{2\pi} e^{im\theta} \frac{e^{i\theta}}{\sqrt{2\pi}}} = \int_{0}^{2\pi} e^{i(m-n\theta)} \theta \]

\[ \delta_{mn} = \frac{2\pi}{\int_{0}^{2\pi} \frac{\sin m\theta}{\sqrt{\pi}} \frac{\sin nx}{\sqrt{\pi}}} \quad m \neq 0 \]

\[ = \frac{2\pi}{\int_{0}^{2\pi} \frac{\sin m\theta}{\sqrt{\pi}} \frac{\sin nx}{\sqrt{\pi}}} \quad m = 0 \]

\[ \delta_{mn} = \frac{2\pi}{\int_{0}^{2\pi} \frac{\cos m\theta}{\sqrt{\pi}} \frac{\cos nx}{\sqrt{\pi}}} \quad m \neq 0 \]

\[ = \frac{2\pi}{\int_{0}^{2\pi} \frac{\cos m\theta}{\sqrt{2\pi}} \frac{\cos nx}{\sqrt{2\pi}}} \quad m = 0 \]
This integral \([0,2\pi]\) is also an interval on which any combination of \(\phi\) and \(\psi\) of those functions is periodic.

\[
\phi(x + 2\pi) = \phi(x) \\
\psi(x + 2\pi) = \psi(x).
\]

Thus on this interval

\[
\int_0^{2\pi} \psi(x) \frac{d\phi(x)}{dx} = \int_0^{2\pi} \frac{d}{dx} \left[ \psi(x)\phi(x) \right] - \int_0^{2\pi} \phi(x) \frac{d\psi(x)}{dx}
\]

\[
= \left[ \psi(x)\phi(x) \right]_0^{2\pi} - \int_0^{2\pi} \phi(x) \frac{d\psi(x)}{dx}
\]

\[
= -\int_0^{2\pi} \phi(x) \frac{d\psi(x)}{dx}.
\]

For \(\exp(i nx), \sin nx, \text{ and } \cos nx\), any interval of length \(2\pi\) will work.

In QM, we call such operators self-adjoint or Hermitian, using the notation

\[
p = \frac{\hbar}{i} \frac{d}{dx}.
\]
Then

\[
\int_0^{2\pi} \psi(x) \frac{i}{\hbar} \frac{d}{dx} \phi(x) = \left. \frac{i}{\hbar} \left[ \psi(x) \phi(x) \right] \right|_0^{2\pi}
\]

\[
-\int_0^{2\pi} dx \left[ \frac{i}{\hbar} \frac{d\psi(x)}{dx} \right] \phi(x)
\]

\[
= \int_0^{2\pi} dx \left[ \frac{i}{\hbar} \frac{d\psi(x)}{dx} \right] \phi(x)
\]

In QM, we often belittle the importance of the boundary conditions.

All these complete orthonormal functions

\[
\frac{\sin x}{\sqrt{2\pi}}, \quad \frac{\sin nx}{\sqrt{n\pi}}, \quad \frac{\cos mx}{\sqrt{m\pi}}
\]

are eigenfunctions of the hermitian
operator \[ \frac{d^2}{dx^2} \] with eigenvalue \(-n^2\):

\[ \frac{d^2}{dx^2} e^{inx} = -n^2 e^{inx} \]

\[ \frac{d^2}{dx^2} \cos nx = -n^2 \cos nx \]

\[ \frac{d^2}{dx^2} \sin nx = -n^2 \sin nx. \]

So we have the usual happy confluence of hermiticity, completeness, and orthonormality.

In QM, \( p = \frac{\hbar}{i} \frac{d}{dx} \) is usually applied to the infinite interval \((-\infty, \infty)\), and it is assumed that all wave functions \( \Psi(x), \Phi(x) \) vanish as \( x \rightarrow \pm \infty \). But the \( z \)-component of angular momentum \( L_z \) is often represented as

\[ L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \]

on the interval \([0, 2\pi]\) with all wave functions periodic on that interval.
Sawtooth Wave

Let \( f(x) = x \) on \([-\pi, \pi]\). Note that \( f(x) \) is not periodic on this interval, so we expect trouble. On this interval, by (14.11), we have

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx = 0 \quad \text{since } x \text{ is odd and } \cos nx \text{ is even.}
\]

And by (14.12)

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx = \frac{1}{\pi} \Im \int_{-\pi}^{\pi} x e^{inx} \, dx
\]

\[
= \frac{1}{\pi} \Im \left[ \int_{-\pi}^{\pi} \frac{d}{dn} \frac{e^{inx}}{in} \right]_{-\pi}^{\pi}
\]

\[
= \frac{1}{\pi} \Im \left[ \frac{x e^{inx}}{n} - \frac{e^{inx}}{in^2} \right]_{-\pi}^{\pi}
\]

\[
= \frac{1}{\pi} \Im \left[ \frac{\pi e^{i\pi n}}{n} + \frac{-\pi e^{-i\pi n}}{n} \right] = \frac{1}{\pi} \Im \left( (-1)^n \frac{2\pi n}{n} \right)
\]

\[
= 2(-1)^{n+1}
\]
So by (14.1)
\[ f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \]
\[ = \sum_{n=1}^{\infty} \frac{2 (-1)^{n+1}}{n} \sin nx \]
\[ = 2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \ldots \right] \]

As more terms are included, the fit improves except near \( x = \pi \) and \( x = -\pi \).

Since \( \sin n\pi = 0 \), the partial sums
\[ f_N(x) = \sum_{n=0}^{N} \frac{2 (-1)^{n+1}}{n} \sin nx \]
all vanish at \( x = \pi \):
\[ f_N(\pi) = 0 \]. Note the overshoot at \( x = \pi - \epsilon \). This does not go away as \( N \to \infty \).
The overshoot near the discontinuity at \( x = \pi \) is an example of the Gibbs phenomenon. More later.

Note that \( f(\pi - \epsilon) = \pi \) while

\[
f(\pi + \epsilon) = -\pi
\]

at least if we imagine that \( x = \pi + \epsilon \) is the same point as \( x = -\pi + \epsilon \). (Back to the original circle of the Laurent series.)

Note that the average of \( f(\pi - \epsilon) \) and \( f(\pi + \epsilon) \) is zero which is the value of all the partial sums \( f_n(x) \) at \( x = \pi \):

\[
f_n(\pi) = \frac{1}{2} \left[ f(\pi - \epsilon) + f(\pi + \epsilon) \right]
\]

\[
= \frac{1}{2} \left[ \pi - \epsilon + \pi + \epsilon \right] = 0,
\]

At such finite discontinuities, a Fourier series converges to the average of the two limits

\[
f_n(x_0) \to \frac{1}{2} \left[ f(x_0 + \epsilon) + f(x_0 - \epsilon) \right].
\]
Sometimes it's easy to sum a Fourier series:

\[
\sum_{n=0}^{\infty} \frac{e^{inx}}{n!} = \sum_{n=0}^{\infty} \left( \frac{e^x}{m!} \right)^n = e^{e^x}.
\]

At other times more effort is required.

Problem 14.1.6:

We use \(-\ln(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \) on the series:

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(x)}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{e^{inx} - e^{-inx}}{2n} \right)
\]

\[
= \sum_{n=1}^{\infty} \frac{i}{2} \left( \frac{(-e^{ix})^n}{n} \right) - \frac{i}{2} \sum_{n=1}^{\infty} \left( \frac{-e^{ix}}{n} \right)^n
\]

\[
= -\frac{i}{2} \ln(1 + e^{ix}) + \frac{i}{2} \ln(1 + e^{-ix})
\]

\[
= \frac{i}{2} \ln \frac{1 + e^{-ix}}{1 + e^{ix}} = \frac{i}{2} \arg \left( \frac{1 + e^{-ix}}{1 + e^{ix}} \right)
\]

\[
= \frac{i}{2} \arg \frac{1 + \cos x - i \sin x}{1 + \cos x + i \sin x} = \frac{i}{2} \arg(1 + \cos x - i \sin x)
\]

\[
= \arctan \left( \frac{\sin x}{1 + \cos x} \right) = \frac{x}{2}, \text{ since } \tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}.
\]
Problem 14.1.2

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(mx - \theta_n) \]

\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left( \cos mx \cos \theta_n + \sin mx \sin \theta_n \right) \]

\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \theta_n \cos mx + \sum_{n=1}^{\infty} a_n \sin \theta_n \sin mx \]

\[ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos mx + b_n \sin mx \]

\[ a_n = a_n \cos \theta_n \quad b_n = a_n \sin \theta_n \]

\[ a_n^2 + b_n^2 = a_n^2 \]

\[ \tan \theta_n = \frac{b_n}{a_n} \]

The coefficients \( a_n \) are the power spectrum, which is invariant under a change in the phases \( \theta_n \).
Fourier series are ideal for periodic functions, but they also work for discontinuous functions.

Consider the interval $[-\pi, \pi]$.

If $f(x)$ is even, then

$$f(x) = a_0 \pi + \sum_{n=1}^{\infty} a_n \cos nx,$$

while if $f(x)$ is odd,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The integrals defining $b_n$ for $f(x)$ even and $a_n$ for odd $f(x)$ vanish.

Say we are given

$$f(x) = x \quad \text{on} \quad 0 \leq x < \pi.$$  

What is $f(x)$ for other values of $x$? 

1) If we use a Taylor expansion, we get

$$f(x) = x \quad \text{everywhere}.$$

2) If we use a cosine series, which makes $f(x)$ even, we get

$$f(x) = \begin{cases} 
-x & -\pi < x \leq 0 \\
2\pi - x & \pi < x < 2\pi 
\end{cases}$$
3) If we use a sine series, making \( f(x) \) odd, then we get:

\[
\begin{align*}
  f(x) &= x & -\pi < x \leq 0 \\
  f(x) &= x - 2\pi & \pi < x < 2\pi
\end{align*}
\]

![Graph of \( f(x) \)]

That is, (2) gives:

![Graph of \( f(x) \)]

while (3) gives:

![Graph of \( f(x) \)]
Suppose \( f(x) \) is the driving force of the linear ODE

\[
\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(mx - \Theta_n)
\]

Then we can try to solve the simpler equation

\[
\frac{d^2y_n}{dx^2} + a \frac{dy_n}{dx} + by_n = \cos(mx - \Theta_n)
\]

and then use the linearity of the ODE to write

\[
y(x) = \frac{a_0 y_0}{2} + \sum_{n=1}^{\infty} a_n y_n(x)
\]

where \( y_0 \) is the solution of

\[
\frac{d^2y_0}{dx^2} + a \frac{dy_0}{dx} + by_0 = 1
\]

More generally, if we use

\[
L = \sum_{m=0}^{\infty} \begin{cases} 0, & m + 1 \text{ odd} \\ h_m(t) \frac{d^m}{dt^m} & m + 1 \text{ even} \end{cases}
\]

to represent a fairly general linear
differential operator, then the equation

\[ L \cdot x(t) = \sum_{m=0}^{\infty} b_m(t) \frac{d^m x}{dt^m} = f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \]

can be solved for each \( n \)

\[ L \cdot x_n(t) = e^{int} \]

to get the solution

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n x_n(t) \]

since

\[ L \cdot x(t) = L \left( \sum_{n=-\infty}^{\infty} c_n x_n(t) \right) \]

\[ = \sum_{n=-\infty}^{\infty} c_n L x_n(t) \]

\[ = \sum_{n=-\infty}^{\infty} c_n e^{int} = f(t) \]

\( L \) is linear because

\[ \frac{d^m}{dt^m} [a \cdot A(t) + b \cdot B(t)] = a \frac{d^m A(t)}{dt^m} + b \frac{d^m B(t)}{dt^m} \]
Of course, this technique also works for any linear expansion of the driving function $f(t)$.

If

$$f(t) = \sum_n c_n \mathcal{Z}_n(t)$$

and

$$L \times n(t) = \mathcal{Z}_n(t),$$

then

$$x(t) = \sum c_n \times n(t)$$

satisfies

$$L \times x(t) = f(t)$$

for an arbitrary linear operator $L$ and any arbitrary basis functions $\mathcal{Z}_n(t)$. 
The original Fourier series over the interval \((-\pi, \pi)\) in terms of sines and cosines is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{14.1}
\]

with

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

Now we let \(x = \frac{\pi t}{L}\) or \(t = \frac{Lx}{\pi}\) and get

\[
F(t) = f(x) = f\left(\frac{\pi t}{L}\right)
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi nt}{L} + b_n \sin \frac{\pi nt}{L}
\]

where

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-L}^{L} F(t) \cos \frac{\pi nt}{L} \left(\frac{\pi}{L}\right) dt
\]

\[
= \frac{L}{L} \int_{-L}^{L} F(t) \cos \left(\frac{\pi nt}{L}\right) dt \quad \text{and similarly}
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} F(t) \sin \left(\frac{\pi nt}{L}\right) dt.
\]
The Square Wave — an example of a discontinuous function

\[ f(x) = 0 \quad -\pi < x < 0 \]

\[ f(x) = h \quad 0 < x < \pi \]

\[ a_0 = \frac{1}{\pi} \int_{0}^{\pi} h \, dt = \frac{h}{\pi} \]

\[ a_n = \frac{1}{\pi} \int_{0}^{\pi} h \cos (nt) \, dt = \frac{2h}{n\pi} \left[ \sin nt \right]_0^\pi = 0 \]

These terms vanish because \( f(x) - \frac{h}{2} \) is an odd function of \( x \) on \((-\pi, \pi)\).

\[ b_n = \frac{1}{\pi} \int_{0}^{\pi} h \sin(nx) \, dt = \frac{h}{n\pi} (1 - \cos n\pi) \]

\[ = \frac{h}{n\pi} (1 - (-1)^n) = \begin{cases} \frac{2h}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \]

So

\[ f(x) = \frac{h}{2} + \frac{2h}{\pi} \left[ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right] \]
because \( f(x) \) is discontinuous, its Fourier coefficients fall off as \( 1/n \) which makes the series conditionally convergent, with big high-frequency components.

The Full-Wave Rectifier

\[
f(x) = \begin{cases} \sin \omega t & -\pi < \omega t < \pi \end{cases}
\]

is even, so no sine terms appear

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \sin \omega t \, dt = \frac{2}{\pi} \int_{0}^{\pi} \sin \omega t \, dt
\]

\[
= \frac{2}{\pi} \left[ \cos \omega t \right]_{0}^{\pi} = \frac{4}{\pi}
\]

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} \sin \omega t \cos n\omega t \, dt
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx = -\frac{4}{\pi} \frac{1}{n^2 - 1}
\]

\( n \) even

\[
a_n = 0 \quad \text{m odd}, \quad \text{so}
\]

\[
f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\omega x}{4n^2 - 1}
\]
Now the continuity of \( f(t) \) has yielded
Fourier coefficients that fall off as \( 1/n^2 \)
and the Fourier series converges absolutely.
So \( f(t) \cong \frac{2}{\pi} \) which is D.C.

More generally, \( a_n \) and \( b_n \) fall off as \( 1/n \)
if \( f(x) \) is discontinuous and as \( 1/n^2 \)
if \( f(x) \) is continuous but possibly with a
discontinuous first derivative.

The Riemann Zeta Function

\[
\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}.
\]

Consider the Fourier series for the even
function \( f(x) = x^2 \) on \( (-\pi, \pi) \).

Then

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}
\]

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx = (-1)^n \frac{4}{n^2}
\]

So

\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \quad -\pi < x < \pi
\]

represents the continuous function
At \( x = \pi \), \( \cos n\pi = (-1)^n \) and so we get

\[
\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} + 4 \zeta(2) \quad \text{or}
\]

\[
\zeta(2) = \frac{1}{4} \cdot \frac{2\pi^2}{3} = \frac{\pi^2}{6}.
\]

Page 878 provides more examples of Fourier series.

**Convergence**

As seen in Sec. 6.6, a uniformly convergent series of continuous functions converges to and defines a continuous function. Sines and cosines are continuous functions, so Fourier series are uniformly convergent only when they represent continuous functions.

But if

(a) \( f(x) \) is continuous on \([-\pi, \pi]\),

(b) \( f(-\pi) = f(\pi) \), and

(c) \( f'(x) \) is piecewise continuous,
then the Fourier series for \( f(x) \) will converge uniformly — that is, 
\[
\lim_{n \to \infty} \left| f_n(x) - f(x) \right| < \varepsilon \quad \text{for} \quad n > N
\]

independently of which \( x \in [\pi, \pi] \).

With or without a discontinuity in \( f(x) \), the Fourier series converges in the mean 
\[ \int_{-\pi}^{\pi} \left| f_n(x) - f(x) \right|^2 \, dx < \varepsilon \quad \text{for} \quad n > N. \]

Integration of Fourier Series

If
\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin nx + b_n \cos nx,
\]

then
\[
\int_{x_0}^{x} f(x') \, dx' = \frac{a_0}{2} (x - x_0) + \sum_{n=1}^{\infty} \left[ a_n \int_{x_0}^{x} \cos nx \, dx' + b_n \int_{x_0}^{x} \sin nx \, dx' \right]_{x_0}
\]

and the extra factors of \( 1/n \) improve the convergence. So Fourier series always may be integrated term by term.
But differentiating makes Fourier series converge less well.

If
\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \]

then
\[ f'(x) = \sum_{n=-\infty}^{\infty} in c_n e^{inx} \]

and
\[ f^{(k)}(x) = \sum_{n=-\infty}^{\infty} (in)^k c_n e^{inx} \]

so the series get worse and worse.