In Section 6.5 of A&W, we saw that a function $f(z)$ analytic in an annulus with center at z_0 possesses the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \tag{6.69}$$

where

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} .$$

The contour C lies in the annulus.

Setting $z_0 = 0$ and choosing the annulus to enclose the unit circle, we have

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$c_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta'})\, i e^{i\theta'}}{e^{i(n+1)\theta'}}\, d\theta'$$

or

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})\, d\theta .$$

By construction, $f(e^{i\theta})$ is periodic in θ

$$f(e^{i(\theta+2\pi n)}) = f(e^{i\theta}) \quad \text{with period } 2\pi.$$

If we use the notation $f(\theta)$ for $f(e^{i\theta})$, then we may write $f(\theta)$ as the Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

with Fourier coefficients a_n

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{-in\theta} f(\theta),$$

as long as we understand that $f(\theta)$ is periodic with period 2π,

$$f(\theta + 2\pi n) = f(\theta).$$

Combining the equations for $f(\theta)$ and a_n, we get

$$f(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{2\pi} \frac{d\theta'}{2\pi} e^{-in\theta'} f(\theta')$$

$$= \int_0^{2\pi} d\theta' \, f(\theta') \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta')}}{2\pi}$$

from which we may infer the completeness relation

$$\delta(\theta-\theta') = \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\theta')}}{2\pi}$$

for the functions $\exp(in\theta)/\sqrt{2\pi}$.

This derivation shows that Fourier-series expansion & the completeness of the functions

$$\langle \theta | n \rangle = \frac{e^{in\theta}}{\sqrt{2\pi}} = F_n(\theta)$$

hold for functions $f(\theta)$ that are related to functions analytic on the unit circle.

Actually they hold for a broader class of functions, including those that are piecewise continuous or square integrable. But convergence is "in the mean" — more later.

This exponential form of Fourier series is simple and complex. The more common form is complex and real. To find it, we express $f(\theta)$ and a_n in terms of their real and imaginary parts:

$$f(\theta) = u(\theta) + i\,v(\theta)$$

$C_n = x_n + i\,y_n$, so that

$$u(\theta) + i\,v(\theta) = \sum_{n=-\infty}^{\infty} (x_n + i y_n)(\cos n\theta + i \sin n\theta)$$

$$= x_0 + iy_0 + \sum_{n=1}^{\infty} (x_n + x_{-n})\cos n\theta - (y_n - y_{-n})\sin n\theta$$

$$+ i \sum_{n=1}^{\infty} (y_n + y_{-n})\cos n\theta + (x_n - x_{-n})\sin n\theta .$$

So $u(\theta)$ has the expansion

$$u(\theta) = x_0 + \sum_{n=1}^{\infty} (x_n + x_{-n})\cos n\theta - (y_n - y_{-n})\sin n\theta$$

and $v(\theta)$ has

$$v(\theta) = y_0 + \sum_{n=1}^{\infty} (y_n + y_{-n})\cos n\theta + (x_n - x_{-n})\sin n\theta,$$

where

$$c_n = x_n + i y_n = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(\cos n\theta - i \sin n\theta\right)\left(u(\theta) + i v(\theta)\right)$$

or

$$x_n = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos n\theta \, u(\theta) + \sin n\theta \, v(\theta)$$

and

$$y_n = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos n\theta \, v(\theta) - \sin n\theta \, u(\theta) .$$

The book's notation is

$$a_n = x_n + x_{-n} = \int_0^{2\pi} \frac{d\theta}{\pi} \cos n\theta \, u(\theta) \qquad (14.11)$$

$$b_n = -(y_n - y_{-n}) = \int_0^{2\pi} \frac{d\theta}{2\pi} \sin n\theta \, u(\theta), \qquad (14.12)$$

whence

$$u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta . \qquad (14.1)$$

If now we set

$$\alpha_n = y_n + y_{-n} = \int_0^{2\pi} \frac{d\theta}{\pi} \cos n\theta \; v(\theta)$$

and

$$\beta_n = x_n - x_{-n} = \int_0^{2\pi} \frac{d\theta}{\pi} \sin n\theta \; v(\theta),$$

then the Fourier series for $v(\theta)$ is

$$v(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\theta + \beta_n \sin n\theta .$$

We have from (14.1) & (14.11-12)

$$u(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \, u(\theta')$$

$$+ \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} d\theta' \left[\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right] u(\theta')$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta' \, u(\theta') + \int_0^{2\pi} d\theta' \, \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n(\theta - \theta')) \; u(\theta')$$

whence

$$\delta(\theta - \theta') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos[n(\theta - \theta')]$$

or

$$\delta(\theta-\theta') = \sum_{n=-\infty}^{\infty} \frac{\cos n(\theta-\theta')}{2\pi} .$$

All these functions, $e^{in\theta}$ and its real and imaginary parts are orthogonal and normalizable on the interval $[0, 2\pi]$.

$$\delta_{nn'} = \int_0^{2\pi} d\theta \left(\frac{e^{in'\theta}}{\sqrt{2\pi}}\right)^* \frac{e^{in\theta}}{\sqrt{2\pi}} = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i(n-n')\theta}$$

$$\delta_{mn} = \int_0^{2\pi} dx \quad \frac{\sin mx}{\sqrt{\pi}} \quad \frac{\sin nx}{\sqrt{\pi}} \qquad m \neq 0$$

$$0 = \int_0^{2\pi} dx \; \frac{\sin 0x}{\sqrt{\pi}} \quad \frac{\sin nx}{\sqrt{\pi}} \qquad m = 0$$

$$\delta_{m,n} = \int_0^{2\pi} dx \quad \frac{\cos mx}{\sqrt{\pi}} \quad \frac{\cos nx}{\sqrt{\pi}} \qquad m \neq 0$$

$$\delta_{0n} = \int_0^{2\pi} dx \; \frac{\cos 0x}{\sqrt{2\pi}} \quad \frac{\cos nx}{\sqrt{2\pi}} \qquad m = 0$$

This interval $[0, 2\pi]$ is also an interval on which any combination of φ; ψ of these functions is periodic

$$\varphi(x + 2\pi) = \varphi(x)$$
$$\psi(x + 2\pi) = \psi(x).$$

Thus on this interval

$$\int_0^{2\pi} dx \; \psi(x) \frac{d}{dx} \varphi(x) = \int_0^{2\pi} dx \frac{d}{dx}\left[\psi(x)\varphi(x)\right] - \int_0^{2\pi} dx \; \varphi(x) \frac{d\psi(x)}{dx}$$

$$= \left[\psi(x)\varphi(x)\right]_0^{2\pi} - \int_0^{2\pi} dx \; \varphi(x) \frac{d\psi(x)}{dx}$$

$$= - \int_0^{2\pi} dx \; \varphi(x) \frac{d\psi(x)}{dx}$$

For $\exp(inx)$, $\sin nx$, and $\cos nx$, any interval of length 2π will work.

In QM, we call such operators self adjoint or hermitian, using the notation

$$p = \frac{\hbar}{i} \frac{d}{dx} \quad .$$

Then

$$\int_0^{2\pi} dx\ \psi^*(x)\ \frac{\hbar}{i}\frac{d}{dx}\varphi(x) = \frac{\hbar}{i}\left[\psi^*(x)\varphi(x)\right]_0^{2\pi}$$

$$-\int_0^{2\pi} dx \left[\frac{\hbar}{i}\frac{d\psi^*}{dx}(x)\right]\varphi(x)$$

$$= \int_0^{2\pi} dx\ \left[\frac{\hbar}{i}\frac{d}{dx}\psi(x)\right]^*\varphi(x)$$

or

$$\langle\psi|\,p\,|\varphi\rangle = \langle\psi|p\varphi\rangle = \langle p\psi|\varphi\rangle.$$

In QM, we often belittle the importance of the boundary conditions.

All these complete orthonormal functions

$$\frac{e^{inx}}{\sqrt{2\pi}}, \quad \frac{\sin nx}{\sqrt{\pi}}, \quad \frac{\cos nx}{\sqrt{\pi}}$$

are eigenfunctions of the hermitian

operator $\frac{d^2}{dx^2}$ with eigenvalue $-n^2$:

$$\frac{d^2}{dx^2} e^{inx} = -n^2 e^{inx}$$

$$\frac{d^2}{dx^2} \cos nx = -n^2 \cos nx$$

$$\frac{d^2}{dx^2} \sin nx = -n^2 \sin nx .$$

So we have the usual happy confluence of hermiticity, completeness, and orthonormality.

In QM, $p = \frac{\hbar}{i}\frac{d}{dx}$ is usually applied to the infinite interval $(-\infty, \infty)$, and it is assumed that all wave functions $\psi(x), \phi(x)$ vanish as $|x| \to \infty$. But the z-component of angular momentum L_z is often represented as

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}$$

on the interval $[0, 2\pi]$ with all wave functions periodic on that interval.

Sawtooth Wave

Let $f(x) = x$ on $[-\pi, \pi]$. Note that $f(x)$ is not periodic on this interval, so we expect trouble. On this interval, by (14.11), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \;\; x\cos(nx) = 0$$ since x is odd and $\cos nx$ is even.

And by (14.12)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \;\; x \sin nx = \frac{1}{\pi}\,\mathrm{Im}\int_{-\pi}^{\pi} dx \; x\, e^{inx}$$

$$= \frac{1}{\pi}\,\mathrm{Im}\; -i \int_{-\pi}^{\pi} dx \;\frac{d}{dn}\, e^{inx} = -\frac{1}{\pi}\,\mathrm{Im}\left[-i\frac{d}{dn}\,\frac{e^{inx}}{in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,\mathrm{Im}\, -i\left[\frac{x e^{inx}}{n} - \frac{e^{inx}}{in^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\,\mathrm{Im}\, -i\left[\frac{\pi e^{in\pi}}{n} + \frac{\pi e^{-in\pi}}{n}\right] = \frac{1}{\pi}\,\mathrm{Im}\, -i\left((-1)^n\frac{2\pi}{n}\right)$$

$$= \frac{2(-1)^{n+1}}{n}$$

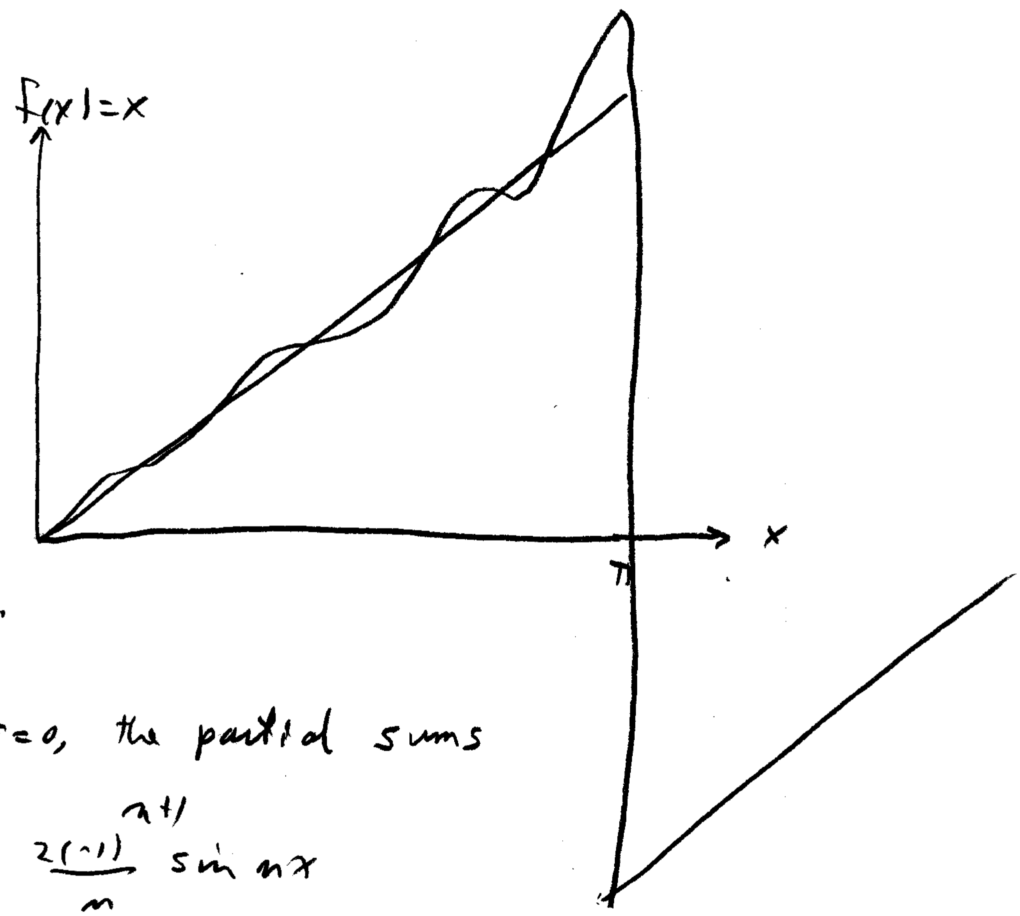
So by (14.1)

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{n} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \cdots\right]$$

Fig 14.1



As more terms are included, the fit improves except near $x = \pi$ and $x = -\pi$.

Since $\sin n\pi = 0$, the partial sums

$$f_N(x) = \sum_{n=0}^{N} \frac{2(-1)^{n+1}}{n} \sin nx$$

all vanish $x = \pi$:

$F_N(\pi) = 0$. Note the overshoot at $x = \pi - \epsilon$. This does not go away as $N \to \infty$.

The overshoot near the discontinuity at $x = \pi$ is an example of the Gibbs phenomenon. More later.

Note that $f(\pi - \epsilon) = \pi$ while

$$f(\pi + \epsilon) = -\pi$$

at least if we imagine that $x = \pi + \epsilon$ is the same point as $x = -\pi + \epsilon$. (Back to the original circle of the Laurent series.)

Note that the average of $f(\pi-\epsilon)$ and $f(\pi+\epsilon)$ is zero which is the value of all the partial sums $f_N(x)$ at $x = \pi$:

$$f_N(\pi) = \frac{1}{2}\left[f(\pi - \epsilon) + f(\pi + \epsilon) \right]$$

$$= \frac{1}{2}\left[\pi - \epsilon + \pi + \epsilon \right] = 0,$$

At such finite discontinuities, a Fourier series converges to the average of the two limits

$$f_N(x_0) \to \frac{1}{2}\left[f(x_0 + \epsilon) + f(x_0 - \epsilon) \right].$$

Sometimes it's easy to sum a Fourier series

$$\sum_{n=0}^{\infty} \frac{e^{inx}}{n!} = \sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!} = e^{e^{ix}}.$$

At other times more effort is required.

Problem 14.1.6:

We use $-\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ on the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{e^{inx} - e^{-inx}}{2ni} \right)$$

$$= \sum_{n=1}^{\infty} \frac{i}{2} \frac{(-e^{ix})^n}{n} - \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-e^{-ix})^n}{n}$$

$$= -\frac{i}{2} \ln(1+e^{ix}) + \frac{i}{2} \ln(1+e^{-ix})$$

$$= \frac{i}{2} \ln \frac{1+e^{-ix}}{1+e^{ix}} = \frac{i}{2} \arg\left(\frac{1+e^{-ix}}{1+e^{ix}} \right)$$

$$= \frac{i}{2} \arg \frac{1+\cos x - i\sin x}{1+\cos x + i \sin x} = i \arg(1+\cos x - i \sin x)$$

$$= \arctan\left(\frac{\sin x}{1+\cos x} \right) = \frac{x}{2}, \text{ since } \tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

Problem 14.1.2

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \left(\cos nx \cos\theta_n + \sin nx \sin\theta_n\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\theta_n \cos nx + \sum_{n=1}^{\infty} \alpha_n \sin\theta_n \sin nx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_n = \alpha_n \cos\theta_n \qquad b_n = \alpha_n \sin\theta_n$$

$$a_n^2 + b_n^2 = \alpha_n^2$$

$$\tan\theta_n = b_n / a_n$$

The coefficients α_n^2 are the <u>power spectrum</u>, which is invariant under a change in the phases θ_n.

Fourier series are ideal for periodic functions, but they also work for discontinuous functions.

Consider the interval $[-\pi, \pi]$.

If $f(x)$ is even, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx ,$$

while if $f(x)$ is odd,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The integrals defining b_n for $f(x)$ even and a_n for odd $f(x)$ vanish.

Say we are given

$$f(x) = x \qquad \text{on} \qquad 0 \leq x < \pi.$$

What is $f(x)$ for other values of x?

1) If we use a Taylor expansion, we get

$$f(x) = x \qquad \text{everywhere}.$$

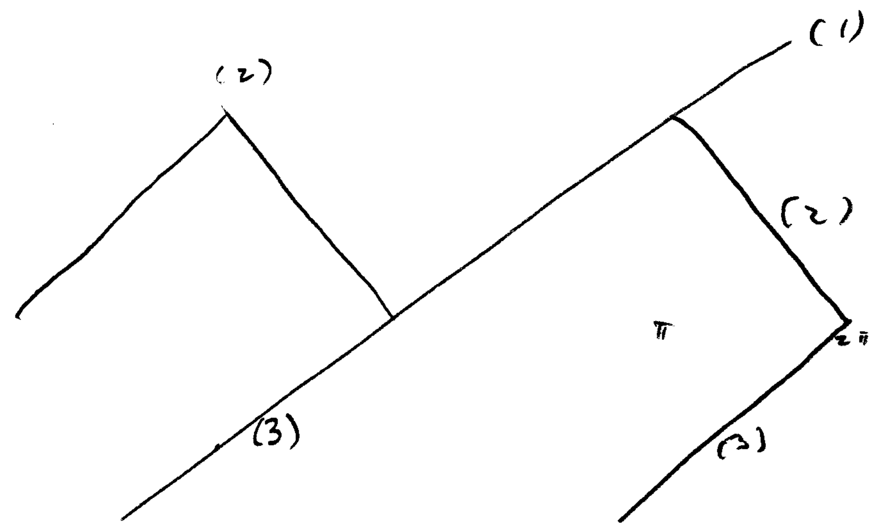
2) If we use a cosine series, which makes $f(x)$ even, we get

$$f(x) = -x \qquad -\pi < x \leq 0$$
$$f(x) = 2\pi - x \qquad \pi < x < 2\pi$$

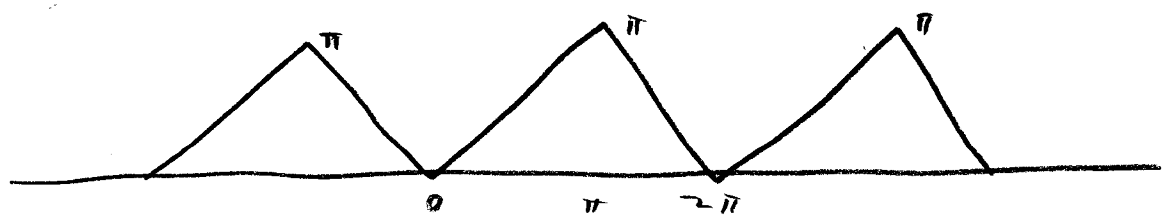
3) If we use a sine series, making $f(x)$ odd, then we get

$$f(x) = x \qquad -\pi < x \le 0$$

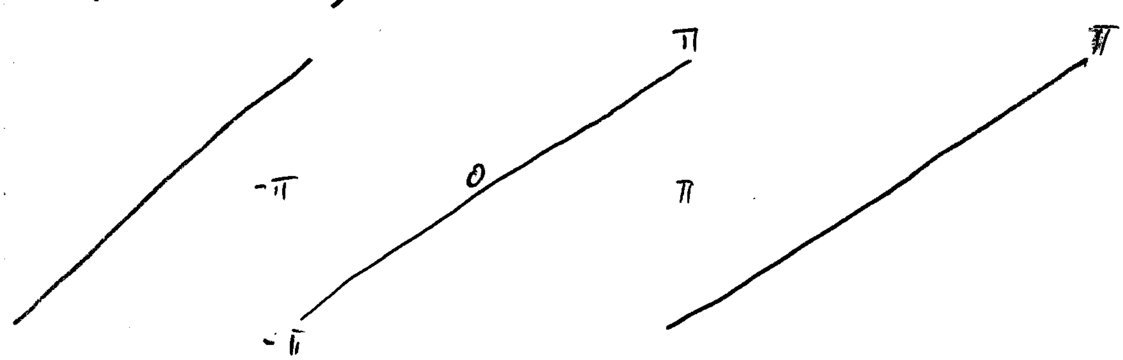
$$f(x) = x - 2\pi \qquad \pi < x < 2\pi$$



That is, (2) gives



while (3) gives



Suppose $f(x)$ is the driving force of the linear ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n)$$

Then we can try to solve the simpler equation

$$\frac{d^2y_n}{dx^2} + a\frac{dy_n}{dx} + by_n = \cos(nx - \theta_n)$$

and then use the linearity of the ODE to write

$$y(x) = \frac{a_0}{2}y_0 + \sum_{n=1}^{\infty} \alpha_n y_n(x)$$

where y_0 is the solution of

$$\frac{d^2y_0}{dx^2} + a\frac{dy_0}{dx} + by_0 = 1 .$$

More generally, if we use

$$L = \sum_{m=0}^{\infty} h_m(t)\frac{d^m}{dt^m}$$

to represent a fairly general linear

differential operator, then the equation

$$L\,x(t) = \sum_{m=0}^{\infty} h_m(t)\,\frac{d^m x}{dt^m} = f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

can be solved for each n

$$L\,x_n(t) = e^{int}$$

to get the solution

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, x_n(t)$$

since

$$\left.\begin{aligned} L\,x(t) &= L \sum_{n=-\infty}^{\infty} c_n\, x_n(t) \\ &= \sum_{n=-\infty}^{\infty} c_n\, L\,x_n(t) \end{aligned}\right\} \text{This works because } L \text{ is linear.}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{int} = f(t),$$

L is linear because

$$\frac{d^m}{dt^m}\left[a\,A(t) + b\,B(t)\right] = a\,\frac{d^m A(t)}{dt^m} + b\,\frac{d^m B(t)}{dt^m}.$$

Of course, this technique also works for any linear expansion of the driving function $f(t)$.

If

$$f(t) = \sum_n c_n Z_n(t)$$

and

$$L x_n(t) = Z_n(t),$$

then

$$x(t) = \sum c_n x_n(t)$$

satisfies

$$L x(t) = f(t)$$

for an arbitrary linear operator L

and any arbitrary basis functions

$Z_n(t)$.

The original Fourier series over the interval $(-\pi, \pi)$ in terms of sines and cosines is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (14.1)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Now we let $x = \pi t/L$ or $t = Lx/\pi$ and get

$$F(t) = f(x) = f\left(\pi \frac{t}{L}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-L}^{L} F(t)\cos \frac{n\pi t}{L}\, \frac{\pi}{L}\, dt$$

$$= \frac{1}{L}\int_{-L}^{L} F(t)\cos\left(\frac{n\pi t}{L}\right) dt$$ and similarly

$$b_n = \frac{1}{L}\int_{-L}^{L} F(t)\sin\left(\frac{n\pi t}{L}\right) dt.$$

The Square Wave — an example of a discontinuous function

$$f(x) = 0 \qquad -\pi < x < 0$$

$$f(x) = h \qquad 0 < x < \pi$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} h\,dt = h$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} h \cos nt \, dt = \frac{-h}{\pi n} \Big[\sin nt \Big]_0^{\pi} = 0$$

These terms vanish because $f(x) - \frac{h}{2}$ is an odd function of x on $(-\pi, \pi)$.

$$b_n = \frac{1}{\pi} \int_0^{\pi} h \sin(nt)\, dt = \frac{h}{n\pi}\left(1 - \cos n\pi\right)$$

$$= \frac{h}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} \frac{2h}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

So

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \frac{\sin(2n+1)x}{2n+1} + \cdots \right].$$

Because $f(x)$ is discontinuous, its Fourier coefficients fall off as $1/n$ which makes the series conditionally convergent, with big high-frequency components.

The Full-Wave Rectifier

$$f(t) = |\sin \omega t| \qquad -\pi < \omega t < \pi$$

is even, so no sine terms appear

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin \omega t| \, d\omega t = \frac{2}{\pi} \int_0^{\pi} \sin \omega t \, d\omega t$$

$$= \frac{2}{\pi} \left[-\cos \omega t \right]_0^{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin \omega t \cos n\omega t \, d\omega t$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx = -\frac{4}{\pi} \frac{1}{n^2 - 1}$$

n even

$a_n = 0$ $\quad n$ odd. So

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\omega t}{4n^2 - 1}$$

Now the continuity of $f(t)$ has yielded Fourier coefficients that fall off as $1/n^2$, and the Fourier series converges absolutely. So $f(t) \approx \frac{2}{\pi}$ which is D.C.

More generally, a_n, b_n fall off as $1/n$ if $f(x)$ is discontinuous and as $1/n^2$ if $f(x)$ is continuous but possibly with a discontinuous first derivative.

The Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} .$$

Consider the Fourier series for the even function $f(x) = x^2$ on $(-\pi, \pi)$.
Then

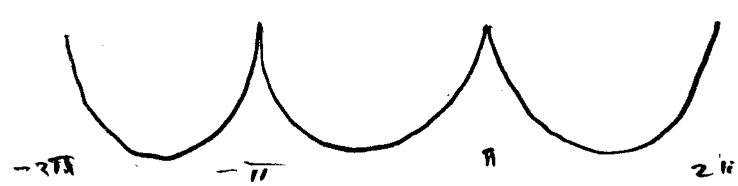
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx = (-1)^n \frac{4}{n^2}$$

So

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2} \qquad -\pi < x < \pi$$

represents the continuous function



At $x = \pi$, $\cos n\pi = (-1)^n$ and so we get

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} + 4\zeta(2) \text{ or}$$

$$\zeta(2) = \frac{1}{4} \frac{2\pi^2}{3} = \frac{\pi^2}{6} .$$

Page 878 provids more examples of Fourier series.

Convergence

As seen in Sec. 5.5 a uniformly convergent series of continuous functions converges to and defines a continuous function. Sines and cosines are continuous functions. So Fourier series are uniformly convergent only when they represent continuous functions.

But if

(a) $f(x)$ is continuous on $[-\pi, \pi]$,

(b) $f(-\pi) = f(\pi)$, and

(c) $f'(x)$ is piecewise continuous,

then the Fourier series for $f(x)$ will converge uniformly — that is,

$$| f_n(x) - f(x) | < \epsilon \quad \text{if} \quad n > N$$

independently of which $x \in [-\pi, \pi]$.

With or without a discontinuity in $f(x)$, the Fourier series converges in the mean, i.e.,

$$\int_{-\pi}^{\pi} | f_n(x) - f(x) |^2 \, dx < \epsilon \quad \text{if} \quad n > N.$$

Integration of Fourier Series

If

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin nx + b_n \cos nx,$$

then

$$\int_{x_0}^{x} f(x')\,dx' = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} -a_n \left[\frac{\cos nx}{n}\right]_{x_0}^{x} + b_n \left[\frac{\sin nx}{n}\right]_{x_0}^{x}$$

and the extra factors of $1/n$ improve the convergence. So Fourier series always may be integrated term by term.

But <u>differentiation</u> makes Fourier series converge less well.

If

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

then

$$f'(x) = \sum_{n=-\infty}^{\infty} in\, c_n e^{inx}$$

and

$$f^{(k)}(x) = \sum_{n=-\infty}^{\infty} (in)^k c_n e^{inx},$$

so the series get worse and worse.