Problem 7.2.17(b)

\[ I = \int_0^\infty \frac{(\ln x)^2}{1 + x^2} \, dx \]

We let \( x = e^y \), \( dx = e^y \, dy \)

\[ I = \int_{-\infty}^{\infty} \frac{y^2 e^y \, dy}{1 + e^{2y}} = \int_{-\infty}^{\infty} \frac{y^2 \, dy}{e^y + e^{-y}} \]

Now \( \int \frac{z^2 \, dz}{e^z + e^{-z}} = \int \frac{z^2 \, dz}{e^z - e^{-z}} + \int \frac{z^2 \, dz}{e^z + e^{-z}} \)

\[ = I + \int_{-R+i\pi}^{-R} \frac{(z+i\pi)^2 \, dz}{e^z + e^{-z}} = I + \int_{-R}^{R} \frac{(z+i\pi)^2 \, dz}{e^z - e^{-z}} \]

\[ = I + I + 2\pi i \int_{-R}^{R} \frac{z \, dz}{e^z - e^{-z}} - \pi^2 \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} \quad (x) \]

The third term vanishes by symmetry, \( \int_{-R}^{R} \frac{z \, dz}{e^z} = 0 \).
Within the contour, the poles are where
\[ e^z - e^{-z} = 0 \quad \text{or} \quad e^z = -1 \]

\[ z = (2m+1)\pi i \]

\[ z = \left(\frac{2m+1}{2}\right)\pi i \]

Only one of these points lies within the contour.

\[ z = \frac{\pi}{2} i \]

So we expand

\[ e^z - e^{-z} = \left(1 + z - \frac{z^2}{2} + \cdots\right) - \left(1 - \frac{z^2}{2} + \cdots\right) \]

\[ = z^2 \left(1 - \frac{\pi^2}{4}\right) + \cdots \]

So the whole contour integral is

\[ \oint \frac{z^2 \, dz}{e^z + e^{-z}} = \oint \frac{z^2 \, dz}{e^z + e^{-z}} \quad \text{around} \ z = \frac{\pi i}{2} \]

\[ z = e^z + \frac{\pi i}{2} \quad dz = i e^z d\theta \]
\[ \int \frac{2z}{z^2} \frac{dz}{2i(z - \frac{m'i}{2})} = \frac{2\pi i}{2i} (\frac{m'i}{2})^2 = -\frac{\pi^3}{4} \]

Similarly, the integral
\[ \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} + \int_{-R+i\pi}^{R+i\pi} \frac{dz}{e^z + e^{-z}} = \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} \]

\[ = \int \frac{dz}{2i(z - \frac{m'i}{2})} = \frac{2\pi i}{2i} = \pi \]

\[ z = \frac{m'i}{2} \]

\[ = \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} - \int_{-R}^{R} \frac{dz}{e^z - e^{-z}} = 2\int_{-R}^{R} \frac{dz}{e^z + e^{-z}} \]

So the fourth term in Eq. (x) is
\[ -\pi^2 \int_{-R}^{R} \frac{dx}{e^x + e^{-x}} = -\pi^2 \frac{\pi}{2} = -\frac{\pi^3}{2} \]

So Eq. (x) says
\[ -\frac{\pi^3}{4} = 2I - \frac{\pi^3}{2} \text{ or } I = \frac{1}{2} \frac{\pi^3}{4} = \frac{\pi^3}{8} \]
Thus

$$\int_{-\infty}^{\infty} \frac{x^2 \, dx}{e^x + e^{-x}} = \int_{0}^{\infty} \frac{(\ln x)^2 \, dx}{1 + x^2} = \frac{\pi^3}{8}.$$
Dispersion Relations

The real part of the complex index of refraction is related to the speed of light in the medium, while the imaginary part is related to the absorption or extinction of the light. Kramers in 1926-27 expressed the real part of $n^{-1}$ as an integral of its imaginary part.

Let us consider a function $f(\beta)$ that is analytic in the UHP and on the real axis, and that vanishes as

$$\lim_{k \to \infty} |f(Re^{i\theta})| = 0 \text{ for } 0 \leq \theta \leq \pi.$$
So for \( z_0 = x_0 + iy_0 \) with \( y_0 > 0 \), we have

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) \, dz}{z - z_0}.
\]

If the contour \( C \) runs along the real axis and then over a great semi-circle of radius \( R \),

\[
z = Re^{i\theta},
\]

then

\[
\left| \oint_C \frac{f(z) \, dz}{z - z_0} \right| \leq \frac{\pi R}{R} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.
\]

So

\[
f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x) \, dx}{x - z_0}, \quad \text{we let} \quad z_0 \rightarrow x_0 \quad \text{in} \quad x, y_0 \rightarrow 0
\]

\[
f(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{-\epsilon} \frac{f(x) \, dx}{x - x_0} + \frac{1}{2\pi i} \int_{\epsilon}^{\infty} \frac{f(x) \, dx}{x - x_0}
\]

\[
+ \frac{1}{2\pi i} \int_{-\epsilon}^{\epsilon} \frac{f(z) \, dz}{z - x_0}
\]
The contour now is

![Contour Diagram]

Recall that \( f(z) \) is analytic in a neighborhood of \( x_0 \), and of every real point.

the first two terms are the Cauchy principal value. The third term is

\[
\frac{1}{2\pi i} \int \frac{f(z)dz}{z-x_0} = \frac{\pi i}{2\pi i} f(x_0) = \frac{f(x_0)}{2}
\]

So

\[
f(x_0) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} dx + \frac{f(x_0)}{2}
\]

or

\[
f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} dx
\]

We let

\[
f(x) = \Re f(x) + i \Im f(x)
\]

\[
= u(x) + i v(x)
\]
Then

\[ u(x_0) + i u(x_0) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \left( \frac{u(x) + i u(x)}{i} \right) \frac{dx}{x-x_0} \]

\[ = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx - \frac{i}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx \]

or

\[ u(x_0) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx \]

and

\[ u(x_0) = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx. \]

These equations are called dispersion relations. The parts \( u(x) \) and \( u(x) \) are also said to be Hilbert transforms of each other.
We often need a symmetry relation like
\[ f(-x) = f^*(x) \]
to make sense of \( u \) and \( v \) for \( x < 0 \).

If
\[ f(-x) = u(-x) + iv(-x) = \sum_{x} \]
then
\[ u(-x) = u(x) \quad \text{even} \]
and
\[ v(-x) = -v(x) \quad \text{odd} \]

Now
\[
U(x_0) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{u(-x)}{x-x_0} \, dx + \frac{1}{\pi} \int_{0}^{\infty} \frac{v(x)}{x-x_0} \, dx
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{0} \frac{-v(-x)}{x-x_0} \, dx + \frac{1}{\pi} \int_{0}^{\infty} \frac{v(x)}{x-x_0} \, dx
\]
\[ y = -x \quad dx = -dy \]
\[
U(x_0) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{u(y)}{-y-x_0} \, dy + \frac{1}{\pi} \int_{0}^{\infty} \frac{v(x)}{x-x_0} \, dx
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{0} \frac{-v(y)}{y+x_0} \, dy + \frac{1}{\pi} \int_{0}^{\infty} \frac{v(x)}{x-x_0} \, dx
\]
So,

\[ u(x_0) = \frac{P}{\pi} \int_{0}^{\infty} \left( \frac{1}{x + x_0} + \frac{1}{x - x_0} \right) u'(x) \, dx \]

\[ = \frac{2}{\pi} \frac{P}{\pi} \int_{0}^{\infty} \frac{x \, u'(x)}{x^2 - x_0^2} \, dx . \]  

Similarly,

\[ u'(x_0) = -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x) \, dx}{x - x_0} \]

becomes

\[ u'(x_0) = -\frac{P}{\pi} \int_{-\infty}^{0} \frac{u(1-x)}{x - x_0} \, dx - \frac{P}{\pi} \int_{0}^{\infty} \frac{u(x)}{x - x_0} \, dx \]

\[ = \frac{P}{\pi} \int_{-\infty}^{0} \frac{u(y) \, dy}{y - x_0} - \frac{P}{\pi} \int_{0}^{\infty} \frac{u(x)}{x - x_0} \, dx \]

\[ = \frac{P}{\pi} \int_{0}^{\infty} \left( \frac{1}{x + x_0} - \frac{1}{x - x_0} \right) u(x) \, dx \]

\[ = -\frac{2}{\pi} x_0 P \int_{0}^{\infty} \frac{u(x)}{x^2 - x_0^2} \, dx . \]
Optical Dispersion

\[ i \omega k^2 - \omega t \]

\[ \epsilon = \omega / k, \quad n = c / \nu = c k / \omega \]

If \( \epsilon \) is the electric permittivity and \( \sigma \) the conductivity, then

\[ k^2 = \epsilon \frac{\omega^2}{c^2} (1 + i \frac{4 \pi \sigma}{\omega \epsilon}) \]

Cgs units

with \( n = 1 \). For \( 4 \pi \sigma / \omega \epsilon \ll 1 \)

\[ \epsilon = \sqrt{\epsilon} \frac{\omega}{c} + i \frac{2 \pi \sigma}{c \sqrt{\epsilon}} \]

\[ i(\omega k - \omega t) \quad i\omega(x \sqrt{\epsilon} / c - t) \quad -2\pi \sigma x / c \sqrt{\epsilon} \]

\[ \epsilon = \epsilon \quad \epsilon \]

\[ m^2 = \frac{c^2 h^2}{\omega^2} = \epsilon + i \frac{4 \pi \sigma}{\omega} \]

Use \( f(\omega) = m^2(\omega) - 1 \rightarrow 0 \) as \( \omega \rightarrow \infty \).

Then K-K had

\[ \text{Re} \left( m^2(\omega_0) - 1 \right) = \frac{2}{\pi} \sum_{0}^{\infty} \frac{\text{Re}(m^2(\omega))}{\omega^2 - \omega_0^2} \]

\[ \text{Im} \left( m^2(\omega_0) \right) = -\frac{2}{\pi} \omega_0 \sum_{0}^{\infty} \frac{\text{Re}(m^2(\omega)) \cdot \omega \left( \text{Re}(m^2(\omega)) - 1 \right)}{\omega^2 - \omega_0^2} \]
Parseval's Relation

We saw that for functions \( f(z) \) analytic in the UMP and on the real axis, if

\[
\lim_{n \to \infty} |f(n e^{i \theta})| \to 0 \quad \text{for} \quad 0 \leq \theta \leq \pi,
\]

then

\[
f(x_0) = \frac{1}{\pi} \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} \, dx. \quad \text{Ann. (7.79)}
\]

So if \( f(z) \) is such a function, we may write

\[
f(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t - x} \, dx \quad \text{(with \( P \) understood)}
\]

and using (7.79) again

\[
f(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{dx}{t - x} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(s) \, ds}{s - x}
\]

\[
= \int_{-\infty}^{\infty} ds \, f(s) \int_{-\infty}^{\infty} \frac{dx}{\pi^2 (t - x)(s - x)}
\]

whence

\[
\delta(t - s) = \frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{dx}{(t - x)(s - x)}
\]

for such analytic functions \( f(z) \).
Delta functions are defined for specific classes of "test" functions — the functions $f(x)$ for which they work.

Now suppose that $u(x)$ and $v(x)$ are the real and imaginary parts of such analytic functions, related by the Hilbert relations:

$$u(x_0) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(x)}{x-x_0} \, dx$$  \quad (7.81a)

and

$$v(x_0) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} \, dx. \quad (7.81b)$$

Suppose also that the integrals

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} |v(x)|^2 \, dx$$

are well defined. Then, using (7.81a) twice, we get

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \int_{-\infty}^{\infty} dx \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(s)}{s-x} \, ds \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(t)}{t-x} \, dt$$

$$= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, 5(t-s) \, v(s) \, v(t)$$

$$= \int_{-\infty}^{\infty} |v(s)|^2 \, ds.$$
Causality requires the effect \( H(t) \) to occur later than its cause \( G(t') \). So we set
\[
F(t-t') = 0 \quad \text{if} \quad t < t',
\]
if
\[
h(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, H(t) e^{i\omega t}
\]
\[
G(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, G(t) e^{i\omega t}
\]
then
\[
h(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, H(t) e^{i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \int_{-\infty}^{\infty} dt' F(t-t') G(t')
\]
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw \, g(w) e^{-i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dt' G(t') e^{i\omega t'}
\]
So, if we accept that the representation
\[ \delta(t'-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega t \delta(t'-t) \]
is appropriate for square-integrable functions, then
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{-i\omega t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t'-t) \]

\[ = \int dt' G(t') \delta(t'-t) = G(t), \]

So
\[ G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{-i\omega t} . \]

Similarly,
\[ F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(w) e^{-i\omega t} \]
and
\[ H(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(w) e^{-i\omega t} . \]
\[ h(\omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dw' f(\omega') e^{\frac{i \omega t'}{\sqrt{2\pi}}} \]
\[ \times \int_{-\infty}^{\infty} dw'' e^{\frac{-i \omega t''}{\sqrt{2\pi}}} g(\omega'') \]

Now using \( \delta \)-nice
\[ S(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{\frac{i \omega t}{2\pi}} \]

we get

\[ h(\omega) = \sqrt{2\pi} \int_{-\infty}^{\infty} dw' \int_{-\infty}^{\infty} dw'' S(\omega - \omega') \delta(\omega' - \omega'') f(\omega') g(\omega'') \]
\[ = \sqrt{2\pi} f(\omega) g(\omega). \]

So the Fourier transform of the convolution
\[ H(t) = \int_{-\infty}^{\infty} F(t - t') C(t') dt' \]

is the product
\[ h(\omega) = \sqrt{2\pi} f(\omega) g(\omega) \] of the Fourier transforms.
Titchmarsh: If \( f(w) \) is square integrable on the real \( w \)-axis, then any one of these three statements implies the other two:

1. The Fourier transform \( \mathcal{F}(t) \) of \( \tilde{f}(w) \) is zero for \( t < 0 \). (This is causality.)

2. \( f(z) \) is analytic in the UH2P for \( y > 0 \) (\( w = z = x + iy \)) and approaches \( f(x) \) almost everywhere as \( y \to 0 \). Also its norm is bounded

\[
\int_{-\infty}^{\infty} |f(x+iy)|^2 dx < K \quad \text{for} \quad y > 0.
\]

3. If \( f = u + iv \), then \( u \) and \( v \) are Hilbert transforms of each other, Eqs. (7.81a–b).

Well, (i) implies that

\[
f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \ \tilde{f}(t) e^{izt}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dt \ \mathcal{F}(t) e^{izt}
\]

is surely analytic in the UH2P, and \( f(z) \) becomes smaller as \( y \) increases. And we have seen that under such conditions \( u \) & \( v \) are related by Hilbert transforms.
Method of Steepest Descent

Consider the integral

$$ I(s) = \int_C f(z) e^{sz} \, dz $$

where \( f(z) \) and \( g(z) \) are analytic functions, as the real variable \( s \) increases.

We may vary the contour \( C \) as long as we collect the residues of any poles that we cross.

This integral will be dominated by the real part of \( f(z) = u(z) + iv(z) \). By the C-R conditions

$$ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}, $$

we have

$$ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = - \frac{\partial^2 v}{\partial x \partial y}.$$ 

So the real part \( u \) of an analytic function is harmonic:

$$ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. $$

Also

$$ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. $$
Since $u$ is harmonic, it can not have any true minima. Instead if

$$\frac{\partial u(z_0)}{\partial x} = 0 = \frac{\partial u(z_0)}{\partial y},$$

then $u(z_0) = u(x_0, y_0)$ has a saddle point at $z_0$. That is, $u$ has a maximum along one curve and a minimum along another curve, both at $z_0 = x_0 + iy_0$.

If $f(z)$ has several saddle points, then we must cope with each of them.

Let's look at one at $z_0$. Then

$$f'(z_0) = 0 \quad \text{and near } z = z_0$$

$$f(z) = f(z_0) + \frac{1}{2} (z - z_0)^2 f''(z_0),$$

Say

$$f''(z_0) = p \ e^{i\phi}. \quad \text{Then we choose our contour through } z_0 \text{ to satisfy } z - z_0 = x e^{i\theta}$$

$$(z - z_0)^2 = x^2 e^{2i\theta}$$

with

$$e^{i\phi} e^{2i\theta} \quad \text{and} \quad e^{i\phi} e^{2i\theta} = -1$$

So

$$\phi + 2\theta = \pi \quad \text{and} \quad \phi + 2(\theta - \pi) = \pi.$$
will describe the straight line thru \( z_0 \). So now \( z \)

\[
f(z) = f(z_0) + \frac{1}{2} x^2 e^{i\theta} p e^{i\phi}
\]

\[
= f(z_0) - \frac{1}{2} x^2 p, \quad \text{so} \quad z = x e^{i\theta}
\]

and

\[
dz = e^{i\theta} dx
\]

\[
\Gamma(s) = e^{i\theta} \int_{-\infty}^{\infty} dx \, q(z) e^{-s f(z)}
\]

\[
= e^{i\theta} \int_{-\infty}^{\infty} dx \, g(z_0) e^{-s f(z_0)}
\]

\[
= e^{i\theta} \int_{-\infty}^{\infty} dx \, g(z_0) e^{-s f(z_0)}
\]

\[
= \sqrt{\frac{2\pi}{s}} \, g(z_0) e^{i\theta} \int_{-\infty}^{\infty} dx \, e^{-s f(z_0)}
\]

\[
= \sqrt{\frac{2\pi}{s e^{-2i\theta}}} \, g(z_0) e^{i\theta} \int_{-\infty}^{\infty} dx \, e^{-s f(z_0)}
\]

Now

\[
pe^{-2i\theta} = pe^{i\phi} = -pe^{i\phi} = -f(z_0),
\]
\[ S_0 \]

\[ I(s) = \sqrt{\frac{2\pi}{-s f''(z_0)}} g(z_0) e^{-5 f(z_0)} \]