First, groups have a finite number of elements. 

A group $G$:
1. $f, g \in G \implies fg \in G$
2. $f(gh) = (fg)h$
3. $\exists e \in G \forall f \in G \implies ef = fe = f$
4. $\forall f \in G, \exists f' \in G \implies f f' = f' f = e.$

A representation of $G$ is a map $D$ onto linear operators $\phi$:
1. $D(e) = \phi_e = I$
2. $D(g_1) D(g_2) = D(g_1 g_2)$.

Example:
$\mathbb{Z}_2 = \{e, a\}$

\[
\begin{array}{c|c|c|c}
 & e & a \\
\hline
\hline
e & e & a \\
\hline
a & a & e \\
\hline
\end{array}
\]

If $g, g_1, g_2 \in G$, then $G$ is abelian.

Rep of $\mathbb{Z}_2$: $D(e) = 1 \quad D(a) = -1.$
This is a 1-D rep of $\mathbb{Z}_2$ because the $D$'s act on a 1-D space.

One may associate with each $g \in G$ a vector $|g\rangle$ and let the $\{|1\rangle, |2\rangle\}$ be an ON basis. Then

$$D(g) |1\rangle |2\rangle = |1\rangle |g12\rangle$$

is the regular rep.

Consider, e.g., $\mathbb{Z}_3$

<table>
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<td>$b$</td>
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<td>$a$</td>
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$|e\rangle = |1\rangle$  
$|e^2\rangle = |a\rangle$  
$|e^3\rangle = |b\rangle$

$$D(e) |e\rangle = <e|D(e) |e\rangle >$$  
$$D(e) |e\rangle = <e|D(e) |e\rangle > = <e|ee\rangle = <e|e\rangle$$

$$= \delta_{i,j} \quad \text{so}$$  
$$D(e) = \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right)$$
$D(a)_{ij} = \langle e_i | D(a) | e_j \rangle$ so

$D(a)_{11} = \langle e_1 | D(a) | e_1 \rangle = \langle e_1 a e \rangle = \langle e_1 a \rangle = \langle e_1 | \mathbf{1} \mathbf{2} \rangle = 0$

$D(a)_{12} = \langle e_1 | D(a) | e_2 \rangle = \langle e_1 a a \rangle = \langle e_1 b \rangle = 0$

$D(a)_{13} = \langle e_1 | D(a) | e_3 \rangle = \langle e_1 a b \rangle = \langle e_1 e \rangle = 1$

Moreover

$D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

This is the regular rep of $S_3$. Another is

$D(e) = 1 \quad D(a) = \mathbf{e} \quad D(b) = \mathbf{e}$
The matrix are matrix elements of the linear operator.

\[ \sigma_{i,j} = \langle e_i | D(g) | e_j \rangle. \]

So

\[ \sigma_{i,j} = \langle \sigma_{i,j} \rangle_{D(g_1) D(g_2)} \]

\[ = \langle e_i | D(g_1) e_i | D(g_2) e_j \rangle \]

\[ = \sum_k \langle D(g_1) e_i | D(g_2) e_j \rangle \]

Irreducible Rep

Similarity transformation

\[ D(g) \rightarrow D'(g) = S^{-1} D(g) S \]

This is another rep, an equivalent rep.

A rep is unitary if all the \( D(g) \) are unitary \( D(g)^* = D(g)^{-1} \) which implies

\( \sigma_{i,j} = \langle e_i | e_j \rangle \)

All reps of finite groups are equivalent to unitary reps.
A rep. is reducible if it has an invariant subspace
\[ PD(g)P = D(g)P \quad \forall g \in G \]

The rep. \( D\) has an invariant subspace projected by
\[ P = \frac{1}{3} \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right) . \]

A rep. is irreducible if it is not reducible.

A rep. is completely reducible if it is equivalent to a rep. like
\[
\begin{pmatrix}
D_1(g) & 0 & 0 \\
0 & D_2(g) & 0 \\
0 & 0 & D_3(g)
\end{pmatrix}
\]

It can be put in block-diagonal form.
A convenient choice of Dirac matrices, used by Weinberg, is

\[ \gamma = -i \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \text{and} \quad \gamma^0 = -i\beta = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \]  

(1)

They satisfy the anti-commutation relations

\[ [\gamma^a, \gamma^b]_+ = 2\eta^{ab}, \]  

(2)

in which the flat space-time metric is

\[ \eta^{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  

(3)

Under hermitian conjugation, they transform as \( \gamma^\dagger = \gamma \) and \( (\gamma^0)^\dagger = -\gamma^0 \). For this choice of Dirac matrices, we may define Majorana and Dirac fields in terms of the scalar-like lawn

\[ \phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3p^0(p^0 + m)}} \left[ \left( \begin{array}{c} I \\ I \end{array} \right) A(p)e^{ipx} + i \left( \begin{array}{c} \sigma_2 \\ -\sigma_2 \end{array} \right) A^*(p)e^{-ipx} \right] \]  

(4)

where \( I \) and \( \sigma_2 \) are the 2 \( \times \) 2 matrices

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]  

(5)

\( A(p) \) and \( A^*(p) \) are the 2-vectors

\[ A(p) = \begin{pmatrix} a(p, +) \\ a(p, -) \end{pmatrix} \quad \text{and} \quad A^*(p) = \begin{pmatrix} a^t(p, +) \\ a^t(p, -) \end{pmatrix}, \]  

(6)

\( p^0 = \sqrt{m^2 + p^2} \), and \( \hbar = c = 1 \). The lawn \( \phi(x) \) describes a single spin-one-half particle that is its own anti-particle.

Since \( m^2 + p^2 = m^2 + p^2 - (p^0)^2 = 0 \), the lawn \( \phi(x) \) satisfies the Klein-Gordon equation

\[ (m^2 + \partial_0^2 - \nabla^2)\phi(x) = (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) = 0. \]  

(7)
The Majorana field \( \chi(x) \) is obtained from derivatives of the lawn \( \phi(x) \):

\[
\chi(x) = (m - \gamma^a \partial_a) \beta \phi(x) .
\]

(8)

It automatically satisfies the Dirac equation:

\[
(\gamma^a \partial_a + m) \chi(x) = (\gamma^a \partial_a + m)(m - \gamma^a \partial_a) \beta \phi(x) \\
= (m^2 - \gamma^a \gamma^b \partial_a \partial_b) \beta \phi(x) \\
= (m^2 - \frac{1}{2} [\gamma^a, \gamma^b] + \partial_a \partial_b) \beta \phi(x) \\
= (m^2 - \eta^{ab} \partial_a \partial_b) \beta \phi(x) \\
= \beta (m^2 - \eta^{ab} \partial_a \partial_b) \phi(x) = 0 .
\]

Suppose that there are two spin-one-half particles of the same mass \( m \) described by the two operators \( a_1(p, \sigma) \) and \( a_2(p, \sigma) \) which satisfy the anti-commutation relations

\[
[a_i(p, \sigma), a_j^\dagger(p', \sigma')] = \delta_{\sigma\sigma'}\delta^3(p - p').
\]

(9)

Then by following Eqs.(4–9) and defining two 2-vectors \( A_i(p, \sigma) \) as in (6), we may construct the two lawns

\[
\phi_i(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0 + m)}} \left[ \left( \begin{array}{c} I \\ I \end{array} \right) A_i(p) e^{ipx} + i \left( \begin{array}{c} \sigma_2 \\ -\sigma_2 \end{array} \right) A_i^*(p) e^{-ipx} \right]
\]

(10)

and from them the two Majorana fields

\[
\chi_i(x) = (m - \gamma^a \partial_a) \beta \phi_i(x)
\]

(11)

which satisfy the Dirac equation

\[
(\gamma^a \partial_a + m) \chi_i(x) = 0.
\]

(12)

But because the two lawns \( \phi_i(x) \) are of the same mass, we may combine them into the complex lawn

\[
\Phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)] .
\]

(13)

From the complex operators

\[
a(p, \sigma) = \frac{1}{\sqrt{2}} [a_1(p, \sigma) + ia_2(p, \sigma)]
\]

(14)
and
\[ a^c(p, \sigma) = \frac{1}{\sqrt{2}} [a_1(p, \sigma) - i a_2(p, \sigma)], \]  
(15)
we may form the complex 2-vectors
\[ A(p) = \frac{1}{\sqrt{2}} [A_1(p) + i A_2(p)] = \begin{pmatrix} a(p, +) \\ a(p, -) \end{pmatrix}, \]  
(16)
and
\[ A^c(p) = \frac{1}{\sqrt{2}} [A_1(p) - i A_2(p)] = \begin{pmatrix} a^c(p, +) \\ a^c(p, -) \end{pmatrix}. \]  
(17)
The complex lawn involves \( A(p) \) and
\[ A^c(p) = \frac{1}{\sqrt{2}} [A_1(p) - i A_2(p)]^* = \frac{1}{\sqrt{2}} [A_1^*(p) + i A_2^*(p)] = \begin{pmatrix} a^c(p, +) \\ a^c(p, -) \end{pmatrix}, \]  
(18)
in the form
\[ \Phi(x) = \int \frac{d^3p}{2 \sqrt{(2\pi)^3} p^0(p^0 + m)} \left[ \begin{pmatrix} I \\ I \end{pmatrix} A(p)e^{ipx} + i \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A^c(p)e^{-ipx} \right]. \]  
(19)
The Dirac field is then
\[ \psi(x) = (m - \gamma^a \partial_a) \beta \Phi(x) \]
\[ = (m - \gamma^a \partial_a) \beta \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)] \]
\[ = \frac{1}{\sqrt{2}} [\chi_1(x) + i\chi_2(x)]. \]
It satisfies the Dirac equation
\[ (\gamma^a \partial_a + m) \psi(x) = 0 \]  
(20)
because the Majorana fields \( \chi_1 \) and \( \chi_2 \) do.

We have defined Majorana and Dirac fields in terms of Weinberg's choice of Dirac matrices. If one uses a different set of Dirac matrices
\[ \gamma'^a = S \gamma^a S^{-1}, \quad \beta' = S \beta S^{-1}, \]  
(21)
then the fields and lawns should be multiplied from the left by the nonsingular matrix \( S \):
\[ \Phi'(x) = S\Phi(x), \quad \psi'(x) = \psi(x), \quad etc. \]  
(22)
Back to continuous groups $G$, Lie groups.

Say $g(x) \in G$ is labeled by the parameters $x_a$ for $a = 1, \ldots, N$ with

$$g(0) = e$$

the identity element.

Suppose $D(x)$ is a representation as linear operators on matrices with

$$D(0) = 1.$$ 

Expand $D(x)$ near 1

$$D(e) = 1 + i \varepsilon_a X_a + \cdots$$

so that

$$X_a = -i \left. \frac{\partial D(x)}{\partial \varepsilon_a} \right|_{\varepsilon = 0}.$$ 

These are the generators of the Lie algebra.

The exponential parametrization of the group is

$$D(x) = \lim_{k \to \infty} (1 + i \varepsilon_a X_a/k)^k = e^{i \varepsilon_a X_a}.$$
Note that

$$D(\frac{i}{\lambda} x_a x_a) = e$$

then

$$D(\lambda) D(\lambda) = D(\lambda^2)$$

because $x_a x_a$ commutes with itself. So

in general,

$$e^{i \alpha x_a} e^{i \beta b x_b} = e^{i (\alpha + \beta) x_a}$$

Instead

$$i \alpha x_a e^{i \beta b x_b} = i \beta b x_b e^{i \alpha x_a}$$

and one may show for small $\alpha, \beta$, that

$$i \beta b x_b = i \alpha x_a + i \beta b x_b - \frac{i}{2} [\alpha x_a, \beta b x_b] + \cdots$$

So

$$[x_a, x_b] = i \delta_{ab} x_c$$

with

$$\delta_{ab} = -\delta_{ba},$$

If the $x_a$'s are Hermitian, then $e$ for real $x_a$ are unitary and

$$i \alpha x_a$$
\[ [X_a, X_b]^+ = -i f^{abc} X_c \]
\[ = [X_b, X_a] = i f^{bac} X_c = -i f^{abc} X_c \]

So \( f^{abc} = f^{bac} \) — the structure constants are real for unitary reps. of Lie groups.

Note that

\[ i S_a X_a = i(\omega_a + \beta a) X_a = -\frac{i}{2} i \omega_a \beta_b f^{abc} X_c \]

In fact the \( f \)'s determine the chain of group multiplication at least when \( D(0) = I \).

The generators \( X_a \) satisfy the Jacobi identity:

\[ [X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0 \]

The matrices \( T^b_a = i f^{abc} \) are the generators in the adjoint rep.

\[ [T^a, T^b] = i f^{abc} T^c \]

Since the \( f \)'s are real, the \( T^a \)'s are purely imaginary,

\[ T^a = -T^a. \]
The $f's$ determine the multiplication law in their representation

$$D(x) D(y) = D(x y),$$

but since

$$D(g_1) D(g_2) = D(g_1 g_2),$$

the only real change when the $f's$ are changed is in the parameters within $g_1 = g(x), \lambda x, \mu x$. If $X' = L B X$, then

$$[X', X''] = L B [X, X']$$

$$= i L B F_{bc} X$$

$$= i L B L_{bc} F_{de} L_{gh} L_{bc} X e$$

$$= i L B L_{bc} F_{de} L_{gh} X'$$

So

$$F_{abc} = L B L_{bc} F_{de} L_{gh} L_{bc} X,$$

The new adjoint map then is

$$(T_a')_{bc} = L B L_{bc} (T_a)_{de} L_{gh} L_{bc} X,$$

$$T_a' = L B L T a L.$$
Now
\[ \text{Tr } T_a T_b \rightarrow \text{Tr } T_a' T_b' \quad \text{and} \]
\[ \text{Tr } (T_a' T_b') = \text{LacLbd } \text{Tr } (L T_c' L^T L T_d' L^T) \]
\[ = \text{LacLbd } \text{Tr } (T_c T_a) \]

Now the matrix
\[ S_{a,b} = \text{Tr } (T_c T_a) = \text{Tr } (T_a T_c) = S_{a,c} \]
is real and symmetric. So
\[ S'_{a,b} = \text{tr } (T_a' T_b') \]
is
\[ S' = L S L^T \]

So we can make \( S' \) diagonal by making \( L \) orthogonal. So
\[ \text{Tr } T_a' T_b' = h^a S_{a,b}. \]

He may rescale these with another \( L \) (diagonal this time) to get
\[ \text{Tr } T_c T_d = \pm S_{a,b}. \]

If all the signs are +, then the group is
compact. In this case
\[ T_a T_b T_c = \lambda S_{ab} \]

So now, dropping primes,
\[ [T_a, T_b] = i f_{abc} T_c \]
\[ [T_a, T_b] T_c = i f_{abc} T_a T_c T_c = i f_{abc} T_a \]

\[ f_{abc} = -\frac{i}{\lambda} T_v ([T_a, T_b] T_c) \]
\[ = -\frac{i}{\lambda} T_v (T_a T_b T_c - T_b T_a T_c) \]
\[ = -\frac{i}{\lambda} T_v (T_b T_c T_a - T_c T_b T_a) \]
\[ = -\frac{i}{\lambda} T_v [T_b, T_c] T_a = f_{bac} \]

So
\[ f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{acb} = -f_{bca} \]

The $T_i$'s are totally anti-symmetric.
\[ (T^b)^{ac} = i \delta_{abc} = -i f_{abc} = -(T^b)^{ca} \]

So the $T_i$'s are unique and anti-symmetric. They are Lémmertian.
So if the group is compact, then the generators of the adjoint rep are hermitian and the matrices of the adjoint rep are unitary:

\[ D(\alpha) = \exp \left( i \alpha \mathbf{a} \mathbf{T} \right) \]

\[ D^*(\alpha) = \exp \left( -i \alpha \mathbf{a} \mathbf{T} \right) = D(\alpha)^{-1} \]

The dimension of the adjoint rep is the number of generators — the rank of \( \mathbf{a} \) etc.

So compact groups have finite-dimensional unitary representations.

Noncompact groups — like \( SU(2, \mathbb{C}) \) & the Lorentz group — do not.
Simple algebras and groups

An invariant subalgebra is a set of $X_a$'s such that for all $Y_b$ in the whole Lie algebra

$$[X_a, Y_b] = i f_{abc} X_c$$

That is $f_{abc} = 0$ for a $\subset$ subalgebra unless $c \in \subset$ sub algebra.

The whole algebra $\Delta 0$ are trivially invariant subalgebras.

A group with no non-trivial invariant subalgebras is simple.

A group with no non-trivial abelian invariant subalgebras is semi-simple.

The adjoint rep of a simple Lie algebra is irreducible. We know

$$T_\mu (Ta T_b) = \delta_{\mu b}$$

If $D$ is reducible, then $E$ is an invariant subspace, spanned by $|n>$, $n = 1 \ldots k$.

Let $x = k + 1, \ldots N$. Let's skip proof.
The simplest compact lie algebra is $su(2)$

$$[J_i, J_j] = i \epsilon_{ijk} J_k \quad J_i^+ = J_i$$

Let's diagonalize $J_3$. Assume a finite-dimensional rep. with $j$ the highest or $-j$ $J_3$:

$$J_3 |j, \alpha\rangle = j |j, \alpha\rangle$$

where $\alpha$ labels degenerate e.v.'s taken to be on

$$<j, \alpha| j, \beta\rangle = \delta_{\alpha \beta}.$$ Let

$$J_3^+ = \frac{1}{ \sqrt{2j+1} } (J_1 + i J_2), \quad (\text{elevation ops.})$$

Then

$$[J_3, J_3^2] = \pm J_3^2 \quad \text{and} \quad [J_3^+, J^-] = J_3,$$

So $J_3 (m, \alpha\rangle = m |m, \alpha\rangle$, hence

$$J_3 J_3^2 (m, \alpha\rangle = (J_3^+ J_3 - J_3 J_3^+) |m, \alpha\rangle$$

$$= (m \pm 1) J_3^2 |m, \alpha\rangle.$$ State with $J_3' = j + 1 > 0$

$$J_3^+ |j, \alpha\rangle = 0 \quad \forall \alpha.$$
$J^{-1} j, \alpha >$ has $J_3'=j-1$ so we may define

$\langle j^{-1}, \alpha | = \frac{1}{N_j(\alpha)} J^{-1} j, \alpha >$, $N_j$ a norm factor.

So

$N_j(\beta)^* N_j(\alpha) \langle j^{-1}, \beta | j^{-1}, \alpha >$

$= \langle j \beta | J^+ J^{-1} j, \alpha > = \langle j \beta | \bar{E} J^+ j, j, \alpha >$

$= \langle j \beta | J_3 j, \alpha > = j \langle j \beta | j, \alpha > = j \delta_{\alpha \beta}$.

Now we have $N_j(\alpha) = \sqrt{j} = N_j$, so

$\langle j^{-1}, \beta | j^{-1}, \alpha > = \delta_{\alpha \beta}$. Also

$J^+ 1 j^{-1}, \alpha > = \frac{1}{N_j} J^+ J^{-1} j, \alpha > = \frac{1}{N_j} \bar{E} J_3 j, j, \alpha >$

$= \frac{J_3}{N_j} j, \alpha > = \frac{1}{\sqrt{j}} j, \alpha > = \sqrt{j} j, \alpha > = N_j j, \alpha >$.

So $J^+$ raises and lowers $J_3'$ but doesn't change $\alpha$.

$J^{-1} j^{-1}, \alpha > = N_{j-1} j^{-1}, \alpha >$

$J^+ j^{-2}, \alpha > = N_{j-1} j^{-1}, \alpha >$. 

There is a tower of states

\[ J^+ \left| j^- k, \alpha \right> = N_{j^- k} \left| j^- k, \alpha \right> \]
\[ J^- \left| j^- k, \alpha \right> = N_{j^- k} \left| j^- k, \alpha \right> \]

Take the \( N_j \)’s real. Then

\[ N_{j^- k}^2 = N_{j^- k} \left< j^- k, \alpha \right| J^- J^+ J^+ J^- \left| j^- k, \alpha \right> \]
\[ = \left< j^- k, \alpha \left| J^+ J^- \right| j^- k, \alpha \right> \]
\[ = \left< j^- k, \alpha \left| J_3 \right| j^- k, \alpha \right> + N_{j^- k+1}^2 \left< j^- k+1, \alpha \left| j^- k+1, \alpha \right> \]
\[ = j^- k + N_{j^- k+1}^2 \quad \text{a recursion relation} \]

\[ N_j^2 = j \]

\[ N_{j-1}^2 - N_j^2 = j-1 \]
\[ N_{j-k+1}^2 - N_{j-k}^2 = j-k+1 \]
\[ N_{j-k} - N_{j-k+1} = j-k \]

\[ N_{j^- k}^2 = j + j^- k + \cdots + j^- k+1 + j-k \]
\[ = (k+1)j - \sum_{\beta=1}^{k} 2 = (k+1)j - \frac{k(k+1)}{2} \]
\[ = \frac{1}{2} (k+1) (2j-k) \]
Let \( h = j - m \), so that \( J^- |j, m, \alpha> \equiv |m, \alpha> \). Then

\[
N_m = \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)}
\]

We've assumed this rep. is finite dimensional. So,

\[
J^- |j-1, m, \alpha> = 0
\]

\[
N_{j-1} = \langle j-1, m | J^+ J^- | j, m, \alpha> = 0
\]

\[
N_{j-1} = \frac{(2j-k)(j-k)}{2} = 0
\]

So \( 2j-k = 0 \) \( \Rightarrow j = \frac{k}{2} \).

So space breaks up into copies, one for each \( k \), each invariant under \( J_i \). But rep. is assumed to be irreducible. So \( \alpha \) is unique. We drop it.

In fact, all the reps. of \([J_i, J_j] = i \delta_{ij} J_0 \) are of this form.

The usual notation is

\[
|j, m, \alpha> \equiv |J, m, \alpha>
\]

\[
|j, m, \alpha> = |m, \alpha> = |j-k, \alpha>
\]
\[ <j, m' | J^2 | j, m> = m \delta_{m, m'} \delta_{j, j'} \]
\[ <j, m' | J+1 | j, m> = \sqrt{(j + m + 1)(j - m + 1)/2} \delta_{m, m'} \delta_{j, j'} \]
\[ <j, m' | J-1 | j, m> = \sqrt{(j + m)(j - m + 1)/2} \delta_{m, m-1} \delta_{j, j'} \]

The spin \( j \) rep of \( SU(2) \) is
\[ [J^a, J^b]_{\text{reps}} = <j, j+1-k | J^a J^b | j, j+1-k> \]

where rows \& columns go from \( 1 \) to \( 2j+1 \).

Usually we can
\[ [J^a, J^b]_{\text{reps}} = <j, m' | J^a J^b | j, m> \]

\( m', m \) form \( j \to j \) in steps of \( 1 \).

\[ J_{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ J_1^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]
\[ J_2^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \]
\[ J_3^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]
Tensor products

\[ |i, x\rangle = |i\rangle |x\rangle \]

\[ D(g) |i, x\rangle = |j, y\rangle \left[ D_{1,2}(g) \right]_{j y i x} \]

\[ = |j\rangle |y\rangle \left( D_{1}(g) \right)_{j i} \left( D_{2}(g) \right)_{y x} \]

\[ = (|j\rangle D_{1}(g)_{ji})(|y\rangle D_{2}(g)_{y x}) \]

Since for small \( \alpha_a \), \( D(g) = \exp(i\alpha_a J_a) \),

\[ (1 + i\alpha_a J_a) |i, x\rangle = |j, y\rangle \left( 1 + i\alpha_a J_a \right) |i, x\rangle \]

\[ = |j, y\rangle \left( \delta_{ji} s_{yx} + i\alpha_a J_{a, jy i x} \right) \]

\[ = |j, y\rangle \left( \delta_{ji} + i\alpha_a J_{aj i} \right) \left( s_{yx} + i\alpha_a J_{a, y x} \right) \]

So

\[ J_{a, jy i x} = J_{aj i} s_{yx} + \delta_{ji} J_{a, y x} \]

\[ J_{a}^{102} = J_{a}^{1} + J_{a}^{2} \]

\[ J_{a} \left( |i\rangle |x\rangle \right) = \left( J_{a} |i\rangle \right) |x\rangle + |i\rangle \left( J_{a} |x\rangle \right) \]
Our basic ans $J_3$ is diagonal, so

$$J_3 \left( |j, m_j > | j' , m_{j'} > \right) = \delta_{m_j + m_{j'}, j' - m_{j'}} \left( |j, m_j > | j', m_{j'} > \right)$$

**Example:**

$$\frac{3}{2}, \frac{3}{2} > = \frac{1}{\sqrt{3}} \left( 1 \frac{1}{2}, \frac{1}{2} > 1, 1 > \right)$$

is the unique "highest-weight" state $J_3 = \frac{3}{2}$.

Now,

$$J^- \left( \frac{3}{2}, \frac{3}{2} > \right) = J^- \left( |1 \frac{1}{2}, \frac{1}{2} > 1, 1 > \right) = \sqrt{\frac{3}{2}} \left| \frac{3}{2}, \frac{3}{2} > \right.$$ 

$$= J^- \left( 1 \frac{1}{2}, \frac{1}{2} > 1, 1 > + \frac{1}{2}, \frac{1}{2} > J^- \left| 1, 1 > \right.$$ 

$$= \sqrt{\frac{3}{2}} \left( 1 \frac{1}{2}, -\frac{1}{2} > 1, 1 > + \frac{1}{2}, \frac{1}{2} > 1, 0 > \right.$$ 

$$1 \frac{1}{2}, \frac{1}{2} > = \sqrt{\frac{1}{3}} \left( \frac{1}{2}, -\frac{1}{2} > 1, 1 > + \sqrt{\frac{2}{3}} \left| 1 \frac{1}{2}, \frac{1}{2} > 1, 0 > \right.$$ 

**Similarly,**

$$J^- \left( \frac{1}{2}, \frac{1}{2} > \right)$$

**gives**

$$1 \frac{1}{2}, -\frac{1}{2} > = \sqrt{\frac{2}{3}} \left| 1 \frac{1}{2}, -\frac{1}{2} > 1, 0 > + \sqrt{\frac{1}{3}} \left| 1 \frac{1}{2}, \frac{1}{2} > 1, 1 > \right.$$ 

$$1 \frac{1}{2}, -\frac{1}{2} > = \left| 1, 0 > \right.$$ 

The states $\pm \frac{1}{2}, \frac{1}{2} >$ and $\pm \frac{1}{2}, -\frac{1}{2} >$ are

$$\frac{1}{2}, \frac{1}{2} > = \sqrt{\frac{2}{3}} \left( \frac{1}{2}, \frac{1}{2} > 1, 1 > - \sqrt{\frac{1}{3}} \left| \frac{1}{2}, \frac{1}{2} > 1, 0 > \right.$$ 

$$\frac{1}{2}, -\frac{1}{2} > = \sqrt{\frac{1}{3}} \left( \frac{1}{2}, -\frac{1}{2} > 1, 0 > + \sqrt{\frac{2}{3}} \left| \frac{1}{2}, \frac{1}{2} > 1, 1 > \right.$$ 

**Since**

$$J^- \left( \frac{1}{2}, \frac{1}{2} > \right) = \sqrt{\left( \frac{1}{2} + \frac{1}{2} \right)} \frac{1}{2}, -\frac{1}{2} > = \frac{1}{\sqrt{2}} \frac{1}{2}, -\frac{1}{2} >,$$
The phase of \( 1/2, -1/2 \) relative to that of \( 1/2, 1/2 \) is determined by \( J^- \).

A **tensor operator** is a set of operators that transform like an irrep of the algebra.

A spin-\( s \) tensor operator \( O^m \) for \( m = -s, \ldots, s \) transforms as

\[
[ J_a, O^m ] = O^m \hspace{0.5cm} (J^a)^{m'm}.
\]

Ex.

\[
J_a = L_a = \epsilon_{abc} \nu_b p_c
\]

\[
[ J_a, r_b ] = \epsilon_{acd} \hspace{0.5cm} [ r_c p_d, r_b ]
\]

\[
= -i \epsilon_{acd} r_c S_{bd}
\]

\[
= -i \epsilon_{abc} r_c = r_c (J^a)^{\text{adj}}_{cb}
\]

Since

\[
(J^a)^{\text{adj}}_{cb} = i \epsilon_{cab},
\]
\[ J_0 \rho^s_{l,j,m} \alpha > = [ J_0, \rho^s_{l,j,m} \alpha > + \rho^s_{l,j,m} \alpha > \]
\[ = \rho^s_{l',j,m} \alpha > (J_0^s)_{l',l} + \rho^s_{l,j',m} \alpha > (J_0^s)_{l,m} \]

So \( \rho^s_{l,j,m} \alpha > \) transforms as \( s \otimes j \)

For \( a = 3 \), in \( J_3 \)-diagonal reps.,

\[ J_3 \rho^s_{l,j,m} \alpha > = l \rho^s_{l,j,m} \alpha > + m \rho^s_{l,j,m} \alpha > \]
\[ = (l+m) \rho^s_{l,j,m} \alpha > \]

\[ J_{3'} = l+m, \]

One may show that

\[ \sum_{l',m} \rho^s_{l,j',m-l,j} \alpha > \left< \frac{s}{j}, l, m - l; j, m \right> \]

and that

\[ \rho^s_{l,j,m} \alpha > = \sum_{J,j} \left< J, l+m | j, j; l, m \right> k_{Jl} \left| J, l+m \right> \]

\[ \text{generally} \Rightarrow \text{Clebsch-Gordan coefficients} \]

The physics leads to

\[ k_{Jl} \left| J, l+m \right> = \sum_{\beta} k_{\beta} \left| J, 0+m, \beta \right> \]
The $k_{\alpha \beta}$ depend on $\alpha, \beta, 0^5$ and on $\beta$ and $J$.

But $k_{\alpha \beta}$ are independent of $l$ and $m$.

So we need to know $k_{\alpha \beta}$ only for one value of $l+m$. These reduced matrix elements are written as

$$k_{\alpha \beta} = \langle J, \beta | 0^5 | l, \alpha \rangle.$$ 

Thus, the Wigner–Eckart theorem:

$$\langle J, m', \beta | 0^5 | l, m, \alpha \rangle = \delta_{m', l+m} \langle J, l+m | 0^5 | J, m, \alpha \rangle \cdot \langle J, \beta | 0^5 | l, \alpha \rangle$$

reduced $\langle \alpha | \text{phys} | \beta \rangle$ Coeffs.

So if we know any non-zero matrix element of a tensor operator between states of some given $J, \beta$, and $j, \alpha$, we can compute all the others using the algebra.
Cayley's subalgebra is a maximal set of commuting hermitean generators $H_i$ $i = 1, \ldots, m$ in an irreducible unitary rep $D$.

$H_i = H_i^\dagger$ \hspace{1cm} $[H_i, H_j] = 0.$

$tr(\{H_i, H_j\}) = k_D S_{ij}$ \hspace{1cm} $i, j = 1, \ldots, m$

$m$ is rank of algebra.

$H_i \{\mu, x, D\} = \{\mu, x, D\}$

The $\mu$'s $M_i$ are the weights. They are real. $\vec{\mu} = (\mu_1, \ldots, \mu_m)$ is the weight vector.

The $\vec{\mu}$'s are the weights of the adjoint rep.
Now in the adjoint representation, the index that labels the generators also labels the rows and columns of the generators, the states—the vectors—of the adjoint rep. correspond to generators. The vector associated with the generator \( X_\alpha \) one may call
\[
|X_\alpha\rangle \rightarrow X_\alpha
\]
and also
\[
\alpha |X_\alpha\rangle + \beta |X_\beta\rangle = |\alpha X_\alpha + \beta X_\beta\rangle \rightarrow \alpha X_\alpha + \beta X_\beta
\]
Use the scalar product
\[
|X_\alpha|X_\beta\rangle = \chi_1 |X_\alpha \rangle \langle X_\beta| \rightarrow (X_\alpha |X_\beta\rangle \neq 0
\]
Now here's something amusing!
\[
X_\alpha |X_\beta\rangle = |X_\alpha X_\beta|X_\alpha \rangle = |X_c\rangle (T_{\alpha})_{\alpha \beta} = i f_{\alpha \beta \epsilon} |X_\epsilon\rangle
\]
\[
= i f_{\epsilon \alpha \beta} |X_\epsilon\rangle = i f_{\beta \epsilon \alpha} |X_\epsilon\rangle
\]
\[
= |X_\epsilon|X_\beta\rangle = |\epsilon [X_\alpha, X_\beta]|\rangle.
\]
So if \( H_i \) and \( H_j \) are in the Cartan subalgebra, then
\[
H_i |H_j\rangle = |\epsilon [H_i, H_j]|\rangle = 0.
\]
So the states corresponding to the Cartan generators have zero weight vectors, \( \tilde{\omega} = 0 \).
\[ \langle H_i | H_j \rangle = \chi_i^* \, t \, (H_i^* H_j) = \chi_i^* \, t \, (H_i H_j) = \delta_{ij} \]

is the normalization that's convenient.

The other states of the adjoint rep \( 1E_\alpha \) have non-zero weight vectors \( \tilde{\alpha} \)

\[ H_i \, | E_\alpha \rangle = \alpha_i \, | E_\alpha \rangle = | \alpha_i \rangle \, | E_\alpha \rangle \]

\[ [H_i, E_\alpha] \]

\( \tilde{\alpha} \) is real

where \( E_\alpha \) are the generators corresponding to the states \( | E_\alpha \rangle \).

So

\[ [H_i, E_\alpha] = \alpha_i \, E_\alpha \]

Take the adjoint

\[ [E_\alpha^+, H_i] = \alpha_i \, E_\alpha^+ \]

\[ [H_i, E_\alpha^+] = -\alpha_i \, E_\alpha^+ \]

One sets \( E_{-\alpha} = E_\alpha \) \((\text{like } J_- = J_+^+)\)

Normalization

\[ \langle E_{\alpha} | E_{\beta} \rangle = \chi_i^* \, t \, (E_{\alpha}^+ E_{\beta}) = S_{\alpha \beta} \]

\[ = \prod_{i=1}^{m} S_{\alpha_i \beta_i} \]
The weights \( \alpha_i \) of the adjoint reps are called \textbf{roots}, and \( \alpha = (\alpha_1, \ldots, \alpha_m) \) is a \textbf{root vector}.

Just as \([J^+_3, J^-_3] = \pm J^+\) allows one to find the irreducible reps of \( SU(2) \) by using the ladder operators \( J^\pm \), so too one may use \([H_i, E_{\pm \alpha}] = \pm \alpha_i E_{\pm \alpha}\) to find the irreps of any compact Lie group. The first step is to take an ev. \( |\mu, D>\) with weight \( \mu^0\) in rep \( D \)

\[ H_i |\mu, D> = \alpha_i |\mu, D> \]

and note that

\[ H_i E_{\pm \alpha} |\mu, D> = \left( [H_i, E_{\pm \alpha}] + E_{\pm \alpha} H_i \right) |\mu, D> \]

\[ = \left( \pm \alpha_i E_{\pm \alpha} + E_{\pm \alpha} H_i \right) |\mu, D> \]

\[ = \left( \pm \alpha_i E_{\pm \alpha} + E_{\pm \alpha} \mu_i \right) |\mu, D> \]

\[ = \left( \mu_i \pm \alpha_i \right) E_{\pm \alpha} |\mu, D> \].

So the weights \( \mu_i \) are shifted by the roots \( \alpha_i \) just as the \( \mu_i \)'s shift by \( \pm 1 \), the roots of \( SU(2) \).

One may show that

\[ [E_{\alpha}, E_{-\alpha}] = \alpha \cdot \mu \]

which is the analog of \([J^+_3, J^-_3] = J^3\). For more stuff on Lie algebras, see Georgi's book, \textit{Lie Algebras and Particle Physics}. 

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Suppose that \( |e\rangle \) is an e.v. of the Hamiltonian \( H \)

\[
H |e\rangle = e |e\rangle.
\]

Suppose that \( H \) is invariant under the unitary transformation \( U \):

\[
U^+ H U = U^+ U H U = H.
\]

So \( U H U = U H \).

\( U \) may represent a translation or a rotation or a "rotation" of the "internal coordinates." The state \( U |e\rangle \) will also be an e.v. of \( H \) with the same energy \( e \):

\[
H(U |e\rangle) = U H |e\rangle = U e |e\rangle = e U |e\rangle.
\]

The transformation implemented by \( U \) will form a group, \( G \). If the group is compact, then the set of states

\[
U(g) |e\rangle
\]

will span a finite dimensional space of e.v.'s of \( H \) with the same energy \( e \). (Call a basis in this space \( |e,i\rangle \). Then

\[
U(g) |e,j\rangle = \sum_i |e,i\rangle \chi_{ij} U(g) |e,j\rangle.
\]