\[
1 + \gamma = e^{ -i \frac{\pi}{2} \frac{\gamma}{\hbar}} = e^{ -i \frac{\pi}{4} + i 0 \times 5 \hbar \frac{\gamma}{4}} \\
= \frac{1}{\sqrt{2}} \left( 1 + i \sigma_x \right)|\uparrow\rangle = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)|\uparrow\rangle \\
= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} (|\uparrow\rangle + i|\downarrow\rangle) \\
1 - \gamma = e^{ -i \frac{\pi}{2} \frac{\gamma}{\hbar}} = \frac{1}{\sqrt{2}} \left( 1 - i \sigma_x \right)|\uparrow\rangle = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \right)|\uparrow\rangle \\
= \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} (|\uparrow\rangle - i|\downarrow\rangle)
\]

So, the \( e \)-states are:

\[
|\uparrow e\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + i|\downarrow\rangle)
\]

\[
|\downarrow e\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - i|\downarrow\rangle)
\]

How could we have all these states using only real numbers?

The transpose of a matrix \( A \) is \( (A^T)_{ik} = A_{ki} \). So

\[
\langle e| e^\dagger \langle A^T \rangle_{ek} = \langle A | e \rangle_{ek}\langle e| e \rangle_{ek}
\]
The (Hermitian) adjoint of a matrix \( A \) is defined by:

\[(A^*)_{mj} = (A_jm)^* .\]

In Dirac's notation, with \( \mathbf{A} \) a linear operator:

\[\langle e_i | A | e_k \rangle = A_{ik} = (e_i, A e_k) .\]

The adjoint operator \( A^* \) is defined by:

\[\langle e_i | A^* | e_k \rangle = (A e_i, e_k) = \langle A^* e_i, e_k \rangle .\]

Taking complex conjugate, we get:

\[\langle e_k | A^* | e_i \rangle = (A e_k, e_i) = (e_k, A^* e_i) .\]

\[
= \langle (A^*)^* | e_i \rangle .
\]

A unitary operator, like \( \langle e_i | U | e_j \rangle = U_{ij} \) where \( \lbrace e_i \rbrace \) and \( \lbrace \uparrow \downarrow \rangle \) are two ON bases, is one whose adjoint is its converse. So if \( \langle e_i | U^* | e_j \rangle = \delta_{ij} \), then:

\[\delta_{ij} = \langle e_i | e_j \rangle = (U e_i, U e_j) = (e_i, \mathbf{U} \mathbf{U}^* e_j) = (e_i, e_j) .\]

So \( \mathbf{U} \mathbf{U}^* = \mathbf{1} \), as we saw on pages 8 & 9.
A hermitian operator $H$ is one that is invariant under hermitian conjugation.

$$H = H^\dagger \quad H_{ij} = H_{ij}^*$$

$$\langle He_i, e_j \rangle = \langle e_i, He_j \rangle = H_{ij}$$

$$\langle e_j, He_i \rangle^* = H_{ji}^* = H_{ij}.$$ If

$$H He_i \rangle = E_i \langle e_i \rangle$$

Then $|e_i \rangle$ is an eigenvector of $H$ with eigenvalue $E_i$. Note that $H(c|e_i \rangle) = E_i (c|E_i \rangle)$, so $c|E_i \rangle$ is an e-vector with the same e-value $E_i$. Suppose $H |e_k \rangle = E_k |e_k \rangle$ as well.

Normalize these e-vectors of $H$,

$$\langle e_k | H |e_i \rangle = E_i \langle e_k | e_i \rangle$$ and

$$\langle e_i | H |e_k \rangle = E_k \langle e_i | e_k \rangle$$ so

$$\langle e_k | H^\dagger |e_i \rangle = \langle e_k | H |e_i \rangle = E_k^* \langle e_k | e_i \rangle.$$

So

$$0 = (E_i - E_k^*) \langle e_k | e_i \rangle,$$

Set $i = k$:

$$0 = (E_k - E_k^*) \langle e_k | e_k \rangle$$

$$= (E_k - E_k^*)$$

So the e-values of hermitian operators are real.
So too

\[ 0 = (e_i - e_k) \langle e_k | e_i \rangle. \]

So if \( e_i \neq e_k \), then \( \langle e_k | e_i \rangle = 0 \); e-vectors with different e-values are orthogonal. If \( H \) is Hermitian,

So if a Hermitian operator \( H \) has \( N \) eigenvalues \( \vec{e}_i \) all different, then its e-vectors \( |e_i\rangle \) are orthogonal. We may normalize them; once we do, they are ON.

**Example** \( \vec{v}_1 = (0\, 1) \)

\[ \vec{v}_1^* = (0\, 1)^T = (0\, 1) = \overline{\vec{v}_1}, \]

Find e-values and e-vectors of \( \vec{v}_1 \)

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \]

So \( \lambda a = b \) and \( \lambda b = a \). So

\[ \lambda^2 a = \lambda b = a \text{ so } \lambda^2 - 1 = 0 \text{ so } \lambda = \pm 1. \]

If \( \lambda = 1 \), then \( a = b \), and the e-vector is

\[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] when normalized.

If \( \lambda = -1 \), then \( b = -a \) and

\[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \] is the normalized e-vector.
So we have seen that the evecs of a hermitian operator can be chosen to be ON. But how do we know that an n x n hermitian matrix will have n evals and n evecs so that the evecs are complete?

To understand this, we look at determinants, which involve the antisymmetric tensors $\varepsilon_{ij}$, $\varepsilon_{ijk}$, $\varepsilon_{ijkl}$, etc. These tensors are totally antisymmetric; the interchange of any two indices changes the sign of $\varepsilon$. Always $\varepsilon_{12\ldots n} = 1$.

For instance, $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$
$\varepsilon_{11} = -\varepsilon_{11} = 0$, $\varepsilon_{22} = 0$

$\varepsilon_{123} = 1$, $\varepsilon_{213} = -1$
$\varepsilon_{231} = 1$, $\varepsilon_{321} = -1$
$\varepsilon_{312} = 1$, $\varepsilon_{132} = -1$

If any index is repeated, $\varepsilon$ vanishes.
So $\varepsilon_{112} = 0$, $\varepsilon_{122} = 0$ etc.

Note that the cross-product has the component
\[(A \times B)_i = \sum_{jk=1}^3 \varepsilon_{ijk} A_j B_k.\]

So $A \times B = \sum_{i=1}^3 \varepsilon_{ijk} A_j B_k$, which follows from $e_j \times e_k = \varepsilon_{ijk} e_i$.
So \( \vec{A} = \sum_j A_j \hat{e}_j \) and \( \vec{B} = \sum_k B_k \hat{e}_k \),

and \( \hat{e}_j \times \hat{e}_k = \varepsilon_{ijk} \hat{e}_i \), then

\[
\vec{A} \times \vec{B} = \left( \sum_{j=1}^{3} A_j \hat{e}_j \right) \times \left( \sum_{k=1}^{3} B_k \hat{e}_k \right) = \sum_{j,k=1}^{3} A_j B_k \hat{e}_j \times \hat{e}_k = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k \hat{e}_i.
\]

The 2×2 determinant of \( A \) is

\[
|A| = \sum_{i,j=1}^{2} \varepsilon_{ij} A_{1i} A_{2j}.
\]

The 3×3 determinant is

\[
|B| = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} B_{1i} B_{2j} B_{3k}.
\]

The n×n determinant is

\[
|M| = \sum_{i_1, i_2, \ldots, i_n}^{M} \varepsilon_{i_1 i_2 \ldots i_n} M_{1i_1} M_{2i_2} \ldots M_{ni_n}.
\]

(Determinants are hard to compute this way.)

More generally,

\[
\varepsilon_{j_1 j_2 \ldots j_n} |M| = \sum_{i_1, i_2, \ldots, i_n}^{M} \varepsilon_{i_1 i_2 \ldots i_n} M_{j_1 i_1} M_{j_2 i_2} \ldots M_{j_n i_n}.
\]
So the cross-product of \( A \) with \( B \) is the determinant

\[
A \times B = \begin{vmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
A_1 & A_2 & A_3 \\
B_1 & B_2 & B_3
\end{vmatrix}
\]

\[
= \sum_{ijk} \epsilon_{ijk} \hat{e}_i \cdot A_j B_k.
\]

Look at \( 2 \times 2 \) case again.

\[
|A| = \sum_{ij} \epsilon_{ij} A_i A_j.
\]

Let \( A'_{ii} = A_{ii} \) and \( A'_{ij} = A_{2i} + c A_{1j} \). Then

\[
|A'| = \sum_{ij} \epsilon_{ij} A'_i A'_j
\]

\[
= \sum_{ij} \epsilon_{ij} A_i (A_{2j} + c A_{1j})
\]

\[
= (|A| + c \sum_{ij} \epsilon_{ij} A_i A_{1j})
\]

\[
= |A| + c (A_{11} A_{12} - A_{12} A_{11}) = |A|
\]

If we add a multiple of row 1 do row 2, then the determinant does not change.
More generally one may add any multiple of any row to any other row without changing the determinant.

The same applies to columns.

Now back to e-val's and e-vecs.

Let $A$ be an $n \times n$ matrix. The important case is when $A$ is hermitian

$$A^* = A^T = A$$

but let's not assume that now.

Then $\psi$ is an e-vec of $A$ with e-val $\lambda$, if

$$A \psi = \lambda \psi.
$$

In this case

$$(A - \lambda I) \psi = 0.
$$

In terms of components,

$$\sum_{j=1}^{\infty} (A_{ij} - \lambda \delta_{ij}) \psi_j = 0.
$$

Consider the vector $V_j$ with $i$th component

$$V_{ij} = A_{ij} - \lambda \delta_{ij}, \quad \text{so} \quad V_1 = \begin{pmatrix} A_{11} - \lambda \\ A_{21} - \lambda \\ \vdots \\ A_{n1} - \lambda \end{pmatrix},
$$

$$V_2 = \begin{pmatrix} A_{12} - \lambda \\ A_{22} - \lambda \\ \vdots \\ A_{n2} - \lambda \end{pmatrix}, \quad \text{etc.}$$


Then the e-vec equation

\[ 0 = \sum_{j=1}^{n} (A_{ij} - \delta_{ij}) \Psi_j \]

is

\[ 0 = \sum_{j=1}^{n} \Psi_j \bar{V}_j \]

So the \( n \) vectors \( \bar{V}_j \) are linearly dependent. (We assume that not all the \( \Psi_j \) vanish.)

Since at least one \( \Psi_j \neq 0 \), label it \( k \).

Then divide by \( \Psi_k \) to get

\[ 0 = \sum_{j=1}^{n} \frac{\Psi_j}{\Psi_k} \bar{V}_j \]

which says that column 1 of the matrix \( A - \lambda I \)

plus \( \frac{\Psi_k}{\Psi_k} \) times column \( k \), plus \( \frac{\Psi_2}{\Psi_k} \) times column 2,

etc., adds up to zero. But each of these operations does not change the determinant of the \( n \times n \) matrix \( A - \lambda I \).

So

\[ |A - \lambda I| = \begin{vmatrix} V_1 & V_2 & \cdots & V_n \end{vmatrix} = \begin{vmatrix} V_1 V_2 (V_1 + \frac{\Psi_2}{\Psi_k} V_1 + \frac{\Psi_3}{\Psi_k} V_2 + \cdots) V_1 \end{vmatrix} \]

The columns of \( A - \lambda I \)

\[ = \begin{vmatrix} V_1 V_2 & 0 & \cdots & V_n \end{vmatrix} \]

\[ = 0 \]

\[ + \text{ column } k \]
So if \( \lambda \) is an e-val of the \( n \times n \) matrix \( A \), then the determinant of \( A - \lambda I \) is zero:

\[
0 = |A - \lambda I|.
\]

This is not a useful way of computing the e-val \( \lambda \), but it does tell us something important. The equation

\[
0 = |A - \lambda I| = \sum_{i=1}^{n} \epsilon_i, i \neq \lambda (A_{ii} - \lambda) (A_{i1} - \lambda_i) (A_{i2} - \lambda_i) \ldots (A_{in} - \lambda_i)
\]

is a polynomial in \( \lambda \) of the \( n \times n \) degree

\[
0 = |A - \lambda I| = (-1)^n \lambda^n + \ldots + |A| = 0.
\]

This is called the characteristic polynomial.

There is a general theorem, the fundamental theorem of algebra, that says that every \( n \times n \) polynomial in \( \lambda \) has \( n \) roots or solutions as long as \( \lambda \) is allowed to be complex. So we now have the general result that every \( n \times n \) matrix \( A \) has \( n \) e-val which in general are complex.

We know that if \( A = A^* \), there \( n \) e-val are all real.

* We'll prove this when we study complex analysis.
Let's review the logic here. If \( \mathbf{A}^\mathbf{V} = \mathbf{0} \), then
\[
0 = (\mathbf{A} - \mathbf{I}) \mathbf{V} = \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n.
\]
It is also true that if \( 0 = (\mathbf{A} - \mathbf{I}) \mathbf{V} \), then the \( n \) vectors \( \mathbf{V}_i \) are linearly dependent and so \( \mathbf{A}^\mathbf{V} = \mathbf{0} \).

This follows from a general theorem:
The determinant \( |\mathbf{V}| = 0 \) if and only if the \( n \) vectors \( \mathbf{V}_i \) are linearly dependent.

We have seen the "if part." The "only if part" is harder to show in general, but it worked out in books on linear algebra.

In the \( 2 \times 2 \) case, we can show that if \( 0 = |\mathbf{V}| \), then \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are linearly dependent.

\[
0 = \begin{vmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_2 & \mathbf{V}_1 \end{vmatrix} = \mathbf{V}_1 \mathbf{V}_2 - \mathbf{V}_2 \mathbf{V}_1.
\]

So \( \mathbf{V}_2 = \mathbf{V}_2 \mathbf{V}_1 / \mathbf{V}_1 \), so

\[
\mathbf{V}_2 = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_2 \\ \mathbf{V}_1 \mathbf{V}_2 / \mathbf{V}_1 \end{pmatrix} = \frac{\mathbf{V}_2}{\mathbf{V}_1} \mathbf{V}_1 = \frac{\mathbf{V}_2}{\mathbf{V}_1} \mathbf{V}_1.
\]

So

\[
0 = \frac{\mathbf{V}_1}{\mathbf{V}_2} \mathbf{V}_1 - \mathbf{V}_2 \mathbf{V}_2 \text{ or } 0 = \frac{\mathbf{V}_2}{\mathbf{V}_1} \mathbf{V}_1 - \mathbf{V}_2 \mathbf{V}_1,
\]

which says that \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are linearly dependent.

The \( m \times n \) case is worked out in books on linear algebra: If the \( m \times n \) matrix \( \mathbf{V} \) has zero determinant, then its columns \( \mathbf{V}_i \) and rows \( \mathbf{V}_i^T \) are linearly dependent.
So if the $m \times n$ matrix $A - \lambda I$ has a determinant that vanishes, $\det(A - \lambda I) = 0$, then its columns are linearly dependent. So there are coefficients $\psi_j$ such that

$$0 = \sum_{j=1}^{n} (A - \lambda I) j \psi_j \quad \text{or} \quad 0 = \sum_{j=1}^{n} (A_{ij} - \lambda \delta_{ij}) \psi_j = 0.$$ 

So for each of the $n$ roots $\lambda_k$ of the characteristic equation

$$0 = \det(A - \lambda I),$$

there is an eigenvector $\psi_k$ of $A$

$$A \psi_k = \lambda_k \psi_k \quad \text{or} \quad \sum_{j=1}^{n} A_{ij} \psi_j = \lambda_k \psi_i.$$ 

In general, these eigenvectors are not orthogonal and the eigenvalues $\lambda_k$ are not real.

We have seen that if $A$ is hermitian, $A^* = A$, then the eigenvalues $\lambda_k$ are real $\lambda_k = \bar{\lambda}_k$ and the eigenvectors with different eigenvalues are orthogonal

$$0 = \langle \psi_m | \psi_l \rangle \quad \forall m \neq l.$$ 

We may make these degenerate eigenvalues real.
A natural way to do that is to add a hermitian perturbation $V$ to $A$, so that the $n$ova $A + \varepsilon V$ are all different.

Then the $n$ e-vects are orthogonal, and we can normalize them.

$$(A + \varepsilon V) |j, \varepsilon \rangle = \delta_{j,k} |j, \varepsilon \rangle$$

$$\langle j, \varepsilon | k, \varepsilon \rangle = \delta_{j,k}.$$ 

Now let $\varepsilon \rightarrow 0$. As $\varepsilon \rightarrow 0$, we'll still have

$$\delta_{j,k} = \langle j, \varepsilon | k, \varepsilon \rangle.$$ 

The vectors $|k\rangle = |k, \varepsilon \rangle$ are then ON and form a basis for $n$-dimensional space. So they are complete

$$\Gamma = \sum_{k=1}^{n} |k\rangle \langle k|$$

$$\langle j | k \rangle = \delta_{j,k}$$

$$A |j\rangle = \chi_j |j\rangle.$$
But if $A$ is an arbitrary $n \times n$ matrix, it still has $n$ e-vectors and $n$ complex e-rods.

$$A |j\rangle = \lambda_j |j\rangle.$$  

The $\lambda_j$'s are the roots of the characteristic equation

$$0 = |A - \lambda I|.$$

Cayley & Hamilton showed that every $n \times n$ matrix satisfies its characteristic equation.

So if

$$0 = |A - \lambda I| = \sum_{i=0}^{n} \prod_{j \neq i} (\lambda - \lambda_j)^2,$$

$$0 = \sum_{i=0}^{n} b_i A^i.$$

We may see why in the case $A^t = A$ in which the e-vectors $|j\rangle$ are ON and complete

$$<i |j\rangle = \delta_{ij},$$

$$I = \sum_{i=1}^{n} |i\rangle \langle i|,$$

$$A |i\rangle = \delta_{i} |i\rangle,$$

$$\delta_{i} = \delta_{i} |i\rangle.$$
For in this case, the product

\[ P(A) = \prod_{i=1}^{n} (A - \lambda_i) \]

which is independent of the order of the factors annihilates every eigvec of \( A \):

\[ P(A) |j\rangle = \prod_{i=1}^{n} (A - \lambda_i) |j\rangle = 0. \]

But the \( |j\rangle\)'s are ON & complete, so

\[ I = \sum_{j=1}^{n} |j\rangle \langle j| \quad \text{and so} \]

\[ P(A) I = P(A) \sum_{j=1}^{n} |j\rangle \langle j| = 0 \]

means \[ P(A) = \prod_{i=1}^{n} (A - \lambda_i) = 0, \]

which is a polynomial equation of order \( n \) (the characteristic equation for \( A \)).
Normal Matrices

A matrix \( A \) is normal if \([A, A^+] = 0\). A has \( n \) e-val \(s \lambda_k \). Let \( B_k = A - \lambda_k I \). Then

\[
[B_k, B_k^+] = [A - \lambda_k I, A^+ - \lambda_k I] = 0
\]

If \( 1k \geq 1k \) is e-vec of \( A \) with e-val \( \lambda_k \), then

\[
0 = (A - \lambda_k I)1k = B_k 1k.
\]

So

\[
<1k B_k^+ B_k 1k> = <1k B_k 1k> = 0.
\]

\[
So \quad 0 = B_k 1k = (A^+ - \lambda_k^*) 1k.
\]

If \( A1j = \lambda_j 1j \) and \( A1k = \lambda_k 1k \), then

\[
<j 1A1k> = \lambda_k <j 1k>
\]

while

\[
<j 1A = (A^+ 1j)^+ = (\lambda_j^* 1j)^+ = \lambda_j <j 1, \quad so
\]

\[
<j 1A1k> = \lambda_j <j 1k> = \lambda_k <j 1k>
\]

whence

\[
0 = (\lambda_j - \lambda_k) <j 1k>.
\]

The e-vec's of different e-val's are \( \perp \) for normal matrices.
We may make the \( n \) e-vecs \( 1k \) of an \( n \times n \) normal matrix \( A \) ON & complete.

So if \( A \) is \( n \times n \) and

\[
0 = [A, A^+] \quad \text{then}
\]

\[
A [k] = \delta_{ik} [k]
\]

and

\[
<k|j> = \delta_{kj} \quad \text{and}
\]

\[
I = \sum_{j=1}^{n} |j><j|
\]

If \( A^* = A \), then \( [A, A^+] = [A, A^*] = 0 \).

So hermitian matrices are normal,

So are unitary matrices: if \( A^* = A^{-1} \), then

\[
[A, A^+] = [A, A^{-1}] = I - I = 0,
\]

The e-vecs of normal matrices can be chosen to be ON & complete.
SVD

Let $A$ be any $m \times k$ (complex) matrix. Then \exists unitary matrices $U$ and $V$ s.t.

$$A = USV$$

where the matrix $S$ is $m \times k$ and zero except for its diagonal entries $s_i$ which are positive or zero. The matrix $U$ is $m \times n$. The matrix $V$ is $k \times k$. The $s_i$ are the singular values of $A$.

Every $m \times k$ matrix possesses such a singular-value decomposition.

Good linear-algebra packages contain programs that perform SVD's.

If $A$ is $m \times n$, then the SVD is called the polar decomposition

$$A = USV$$

where now all 4 matrices are $m \times n$ and $S$ is diagonal

$$S = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{pmatrix}, \quad s_i \geq 0.$$
If the matrix $A$ has entries $V_{ij}$ and by its unitarity, $VV^* = I$,

$$\sum_{e=1}^{K} V_{ie} V_{ej}^* = \delta_{ij}.$$ 

So the vectors

$$\psi_j = \begin{pmatrix} V_{j1}^* \\ \vdots \\ V_{jk}^* \end{pmatrix}$$

are mapped by $V$ into

$$(V \psi_j)_i = \sum_{e=1}^{K} V_{ie} \psi_{ej} = \sum_{e=1}^{K} V_{ie} V_{ej}^* = \delta_{ij}.$$ 

So

$$\sum_{p}^{K} S_{pi} (V \psi_j)_i = \sum_{p}^{K} S_{pi} \delta_{ij} = S_{pj}$$

$$= S_p \delta_{pj} \quad \forall \ j \leq m$$

$$= 0 \quad \forall \ j > m.$$ 

So $A \psi_j = U S V \psi_j = 0 \quad \forall \ j > m.$

$$\psi_j = \sum_{q=1}^{m} U_{ij} \psi_{qj} S_{qj} = U_{ij} S_j \quad \forall \ j \leq m.$$
\[ \phi_j = (u_{ij}) \]

are the columns of a \( m \times n \) unitary matrix \( U \), then

\[ A \psi_j = 0, \quad \psi_j \in \mathbb{R}^m, \quad j = m \]

\[ A \psi_j = 0, \quad 1 \leq j \leq n. \]

The vectors \( \psi_j \) which are the rows of \( V^* \) are the right singular vectors of \( A \) with singular values \( \sigma_j \).

Similarly, the rows of \( U^* \) are the left singular vectors \( \ell_j \) of \( A \).

\[ (\ell_j)_i = (u^*)_i = u_{ij} \]

\[ (\ell_j A)_{ij} = \sum_{k=1}^{m} \ell_{jk} A_{kq} = \sum_{k=1}^{m} u_{kj} (USV)_{iq} \]

\[ \sigma_j \sum_{i=1}^{m} u_{ij} \sum_{r=1}^{n} (SV)_{rq} = \sum_{r=1}^{n} \sigma_j \ell_{jr} (SV)_{rq} \]

because

\[ U^* U = U U^* = I \]

\[ (U^* U)_{ij} = \sum_{k=1}^{m} u_{ik}^* u_{kj} = \delta_{ij}. \]
\[(L_j A)_{ij} = (SV)_{ij} = \begin{cases} 0 & \text{if } j > k \\ s_j V_{ij} & \text{if } j \leq k \end{cases}\]

Thus \(L_j A\) is the left singular vectors \(A\) with singular value \(s_j\) (if \(j \leq k\))

\[L_j A = s_j V_j\]

or

\[(L_j A)_{ij} = s_j V_{ij} \cdot \]

So \(A\) maps the right \(s\)-vectors

\[R_j = V_j = \begin{pmatrix} V_{ij}^* \\ V_{kj} \end{pmatrix}\]

into the columns of \(V\)

\[A R_j = s_j \begin{pmatrix} k_{ij} \\ a_{n_j} \end{pmatrix},\]

and \(A\) maps the left \(s\)-vectors \(L_j\)

into the rows of \(V\)

\[L_j A = s_j \begin{pmatrix} V_{ij} \\ V_{sk} \end{pmatrix}\]