Chaos

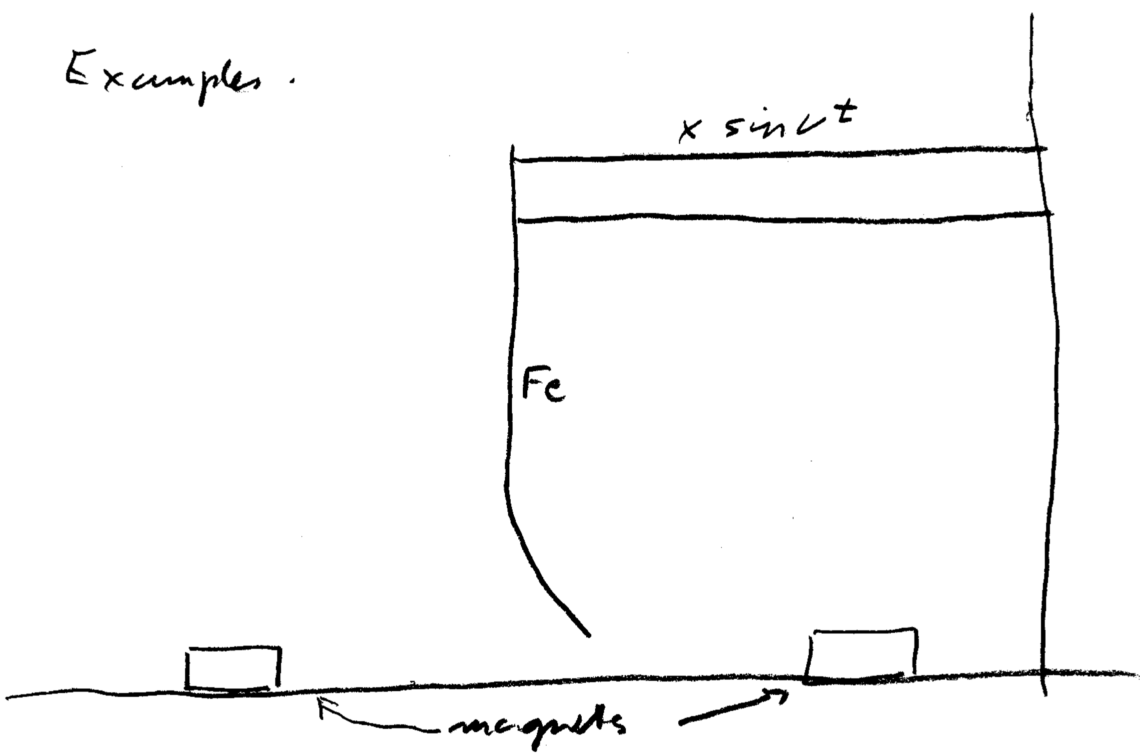
Henri Poincaré (~1900) studied the three-body problem and found very complicated (chaotic) orbits.

There seem to be four kinds of classical motion

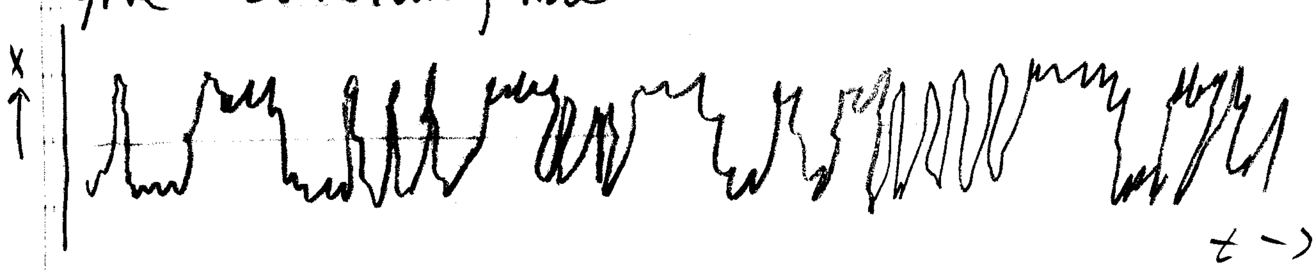
1) periodic
2) steady (or damped motion that stops)
3) quasiperiodic (mixture of periodic motions, ω_i)
4) chaotic

for a system after a transient period.

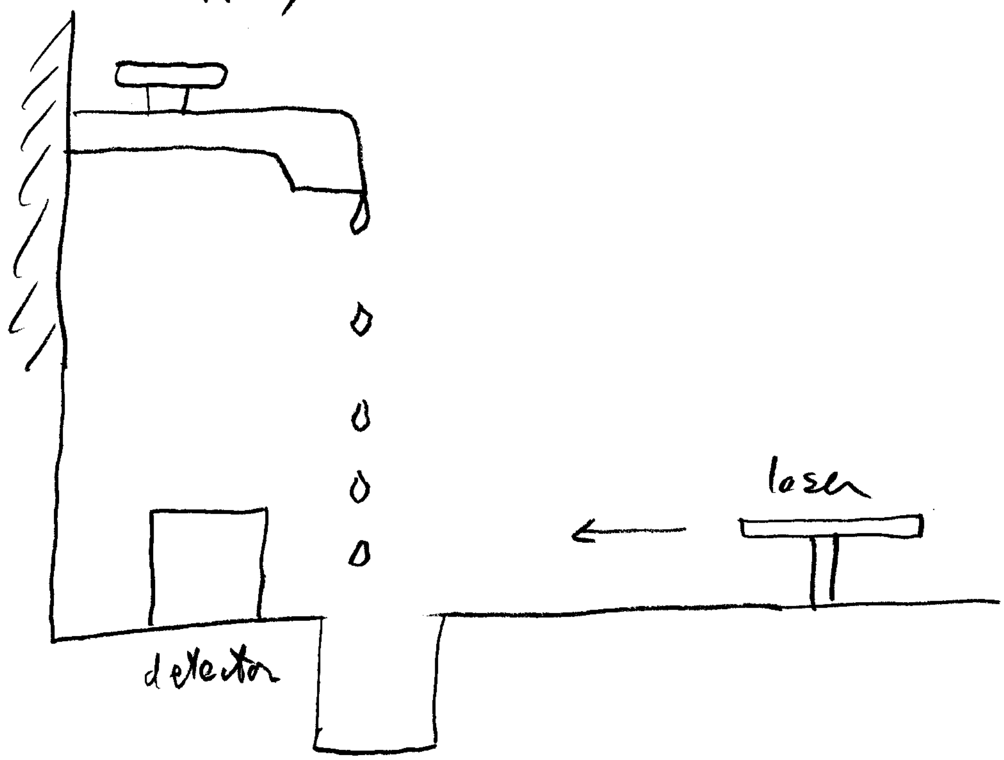
Examples.



$$x'' + \nu \dot{x} + x^3 - x = g \sin t$$ Exp. & theory give something like



Dripping faucet



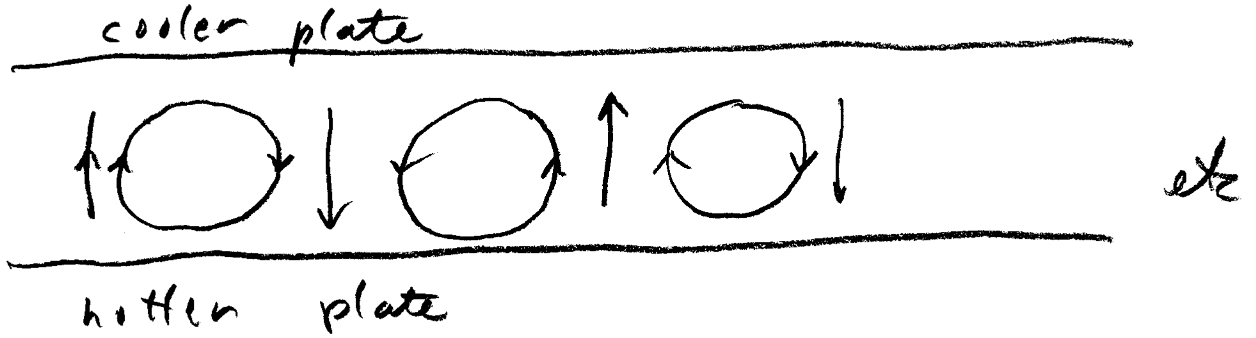
Data are $t_1, t_2, t_3 \quad \cdots$

At low flow rate, $\Delta t_n = t_{n+1} - t_n$ is constant, all Δt_n are equal.

At a slightly higher rate, the drops come with gaps that alternate $\Delta t_a, \Delta t_b, \Delta t_a, \Delta t_b, \cdots$ so that $\Delta t_{n+2} = \Delta t_n$. This is a period-two sequence.

At still higher flow rates, no regularity is apparent.

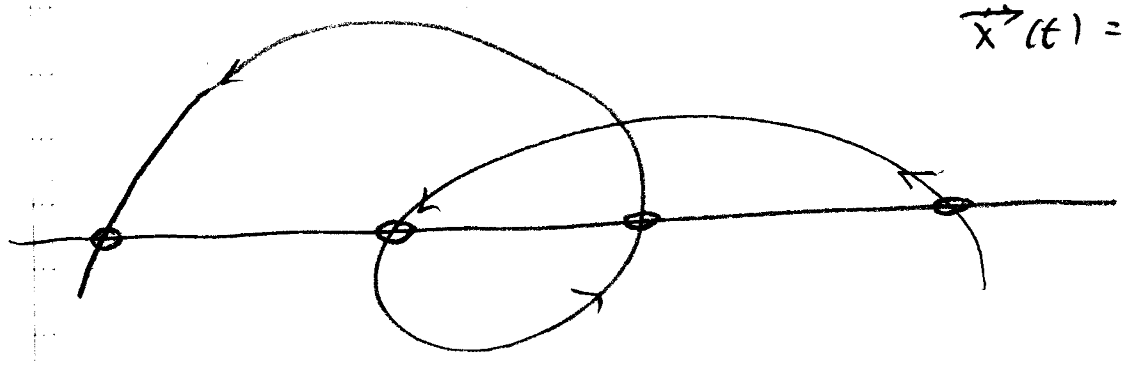
Chaotic Rayleigh - Benard convection occurs when a fluid is placed in a gravitational field between two plates that are kept at constant temperatures with the lower plate hotter by Δt above the chaotic threshold. For lower Δt, the motion is steady convective cellular flow



Dynamical Systems

$$\dot{x}_i = F_i(x) \quad \text{or} \quad \dot{\vec{x}} = \vec{F}(\vec{x}), \text{ i.e.}$$

$$\dot{\vec{x}}(t) = \vec{F}(\vec{x}(t))$$



The crossings of a suitably oriented plane give rise to a map

$$\vec{x}_{n+1} = \vec{M}(x_n)$$

in one fewer dimensions.

In the system

$$\dot{\vec{X}} = \vec{F}(\vec{x})$$

chaos can occur only if the dimension N of the vector $\vec{x}$ exceeds 2

$$N \geq 3.$$

For the invertible map

$$x_{n+1} = M(x_n) \quad \Rightarrow \quad x_n = M^{-1}(x_{n+1})$$

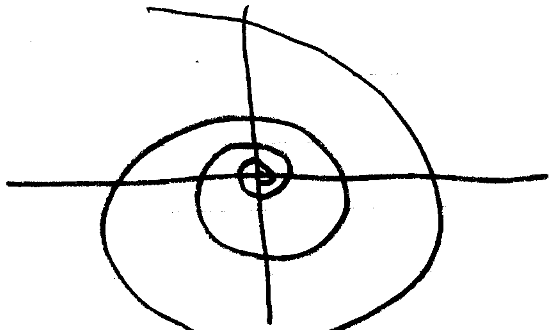
chaos occurs only if $N \geq 2$.

If the map is not invertible, then chaos can occur even if $N = 1$. An example is

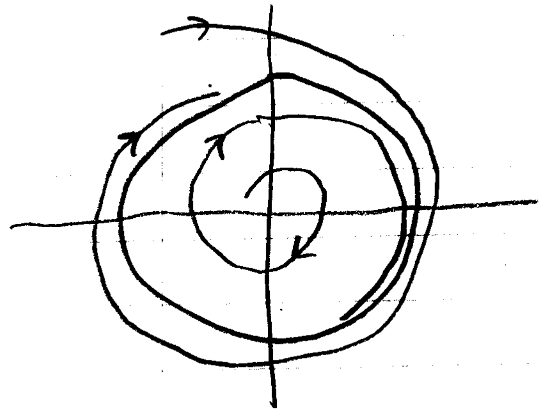
$$x_{n+1} = r\, x_n (x_n - 1)$$

which is not invertible and does exhibit chaos in increasingly striking forms as r exceeds a number slightly greater than $r_m = 3.57$. By $r = 4$, the map is totally chaotic.

Attractors



Here $x_1 = x_2 = 0$ is an attractor.



Here the circle is an attractor called a <u>limit</u> <u>cycle</u>.

The limit cycle occurs in the van der Pol equation

$$\ddot{y} + (y^2 - \eta)\dot{y} + \omega^2 y = 0$$

which may be written as the first-order system

$$x_1 = \dot{y} \qquad x_2 = y$$

$$\dot{x}_1 = -\omega^2 x_2 - (x_2^2 - \eta) x_1$$

$$\dot{x}_2 = x_1 .$$

The van der Pol equation was introduced in the 1920's to describe a vacuum-tube oscillator circuit.

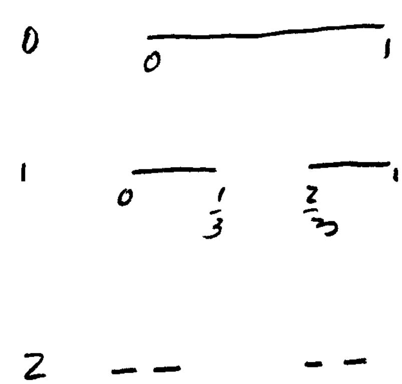
Fractals

Fractal sets don't have dimensions that are natural numbers. To compute their dimensions one needs a definition of dimension.

The box-counting dimension is as follows: Cover the set with line segments, squares, cubes, etc., of edge length ϵ. Count how many you need as $\epsilon \to 0$. Call the number of boxes $N(\epsilon)$. Then

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)},$$

Cantor set:



$\epsilon_n = \left(\frac{1}{3}\right)^n$ need $N(\epsilon) = 2^n$ boxes.

So

$$D_0 = \lim_{n \to \infty} \frac{\ln 2^n}{\ln 3^n} = \lim \frac{\ln 2}{\ln 3} \simeq 0.63.$$

Attractors of fractal dimension are <u>strange</u>.