Consider the ODE

$$y'' - \frac{6}{x^3}\, y = 0 .$$

Since $x^2(-6/x^3)$ diverges as $x \to 0$, the point $x=0$ is an essential singularity of this ODE.

Let

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda \, x^{k+\lambda}$$

$$x^3 y'' = \sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1)\, a_\lambda \, x^{k+\lambda+1} = \sum_{\lambda=0}^{\infty} 6\, a_\lambda \, x^{k+\lambda}$$

The lowest power of x is

$$6 a_0 x^k ,$$

so the indicial equation is

$$a_0 = 0 .$$

But by construction, $a_0 \neq 0$. So we have no solution at all by this series-expansion method.

For the ODE, the point $x=0$

$$x^2 y'' + x y' - a^2 y = 0$$

is a regular singular point.

We try $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

$$\sum_\lambda (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda} + (k+\lambda) a_\lambda x^{k+\lambda} - a^2 a_\lambda x^{k+\lambda} = 0$$

For $\lambda = 0$

$$k(k-1) + k - a^2 = 0$$
$$k^2 = a^2 \quad \text{So} \quad k = \pm a.$$

But then for $\lambda = 1$,

$$\left[(k+1)k + k+1 - a^2\right] a_1 = 0$$

$$\left[k^2 + 2k + 1 - a^2\right] a_1 = (2k+1) a_1$$

$$= (1 \pm 2a) a_1 = 0$$

So $a_1 = 0$ unless $a = \mp 1/2$.

for $\lambda = 2$

$$\left[(k+2)(k+1) + k+2 - a^2\right] a_2 = 0$$

$$= \left(k^2 + 4k + 4 - a^2\right) a_2 = 4(k+1) a_2 = 0$$

So $a_2 = 0$ unless $k = \pm a = -1$. So apart from special values of a, the only solutions are

$$y = x^a \quad \text{and} \quad y = x^{-a}.$$

The ODE

$$x^2 y'' + y' - a^2 y = 0$$

has $x=0$ as an essential singularity.
We try

$$y = \sum a_\lambda x^{k+\lambda}$$

$$\sum (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda} + (k+\lambda) a_\lambda x^{k+\lambda-1} - a^2 a_\lambda x^{k+\lambda} = 0$$

Now the vanishing of the coefficient of the lowest power of x gives

$$0 = k a_0 x^{k-1} \quad \text{or} \quad k = 0$$

$$\sum \left[(k+j)(k+j-1) a_j x^{k+j} + (k+j+1) a_{j+1} x^{k+j} - a^2 a_j x^{k+j} \right] = 0$$

So

$$\left[(k+j)(k+j-1) - a^2 \right] a_j = -(k+j+1) a_{j+1}$$

$$a_{j+1} = a_j \frac{a^2 - j(j-1)}{j+1}$$

So if $a^2 = j(j-1)$, the series terminates, but for general a

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} |j| = \infty$$

and the series diverges for all x.

Fuch's Theorem

We always can obtain at least one power-series solution if we expand about a regular point or a regular singular point.

Do we get a second solution too?

1. If the two roots of the indicial equation are equal, we only get one solution.

2. If the two roots differ by a non-integral number, one gets two solutions.

3. If the two roots differ by an integer, the larger of the two roots yields a solution.

If the only set of numbers $k_1 \cdots k_n$ for which

$$0 = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x)$$

for some range of x is $k_i = 0$, for $i = 1, 2 \cdots n$, then the functions $y_1, y_2, \cdots y_n$ are linearly independent. Otherwise, the y_i are linearly dependent.

If $y_1, y_2, \cdots y_n$ are linearly dependent, then for some $k_1, k_2, \cdots k_n$, we have

$$0 = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x)$$

and so

$$0 = k_1 y_1'(x) + k_2 y_2'(x) + \cdots + k_n y_n'(x)$$

and

$$0 = k_1 y_1''(x) + k_2 y_2''(x) + \cdots + k_n y_n''(x)$$

$$\cdots\cdots$$

$$0 = k_1 y_1^{(n-1)}(x) + k_2 y_2^{(n-1)}(x) + \cdots + k_n y_n^{(n-1)}(x)$$

So if Y is the matrix

$$Y_{ij}(x) = y_j^{(i-1)}(x),$$

then $0 = Y_{ij}(x)\, k_j$ or $Y(x)\, k = 0$.

So the determinant $|Y(x)| = 0$ must vanish if the y_i are linearly dependent.

$$W(x) = |Y(x)|$$

is called the Wronskian.

A Second Solution

Consider the ODE

$$y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = 0.$$

Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions, Then the Wronskian

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1' \neq 0.$$

$$W' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

must satisfy

$$W' = y_1(-P y_2' - Q y_2) - y_2(-P y_1' - Q y_1)$$

$$W'(x) = P(x)\,(y_2 y_1' - y_1 y_2') = -P(x)\,W(x)$$

So we can integrate the Wronskian

$$(\ln W)' = -P$$

$$\ln W(x) = -\int^x dx'\, P(x') + C$$

$$W(x) = W(a)\, e^{-\int_a^x dx'\, P(x')}$$

But $y_1^2 \dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right) = y_1 y_2' - y_2 y_1' = W$

and so

$$\frac{d}{dx}\,\frac{y_2}{y_1} = \frac{W(x)}{y_1^2(x)}$$

whence

$$\frac{y_2(x)}{y_1(x)} = \int^x dx' \,\frac{W(x')}{y_1^2(x')} + c$$

or

$$y_2(x) = y_1(x)\left(\int^x dx' \,\frac{W(x')}{y_1^2(x')} + c\right)$$

$$= y_1(x)\left(\int^x dx' \,\frac{W(a)\,e^{-\int_a^{x'} dx'' P(x'')}}{y_1^2(x')} + c\right)$$

So given one solution, $y_1(x)$, one may generate a second solution $y_2(x)$

$$y_2(x) = y_1(x)\int^x dx' \,\frac{e^{-\int^{x'} dx'' P(x'')}}{y_1^2(x')}$$

apart from additive and multiplicative constants.

An important special case is:

$$P(x) = 0$$

so that

$$y'' + Q \cdot y = 0 ,$$

In this case,

$$W' = -PW = 0$$

and so the Wronskian W is a constant

$$W = y_1 y_2' - y_1' y_2 = c$$

In this case, the general formula for y_2 is just

$$y_2(x) = y_1(x) \int^x dx' \frac{1}{y_1^2(x')} .$$

As long as we expand about a regular point or a regular singular point, we always may use the series method to find $y_1(x)$. Then we may use the Wronskian formula to get a second solution $y_2(x)$. One may also generate a second solution $y_2(x)$ by the series method of pages 538 - 542 of A&W. So a 2d-order linear homogeneous ODE gives two linearly independent solutions in general. Two, but not three.

Why not three?

Suppose y_i for $i = 1, 2, 3$ are three solutions of the 2d-order linear homogeneous ODE

$$0 = y_i'' + P y_i' + Q y_i .$$

Then the 3d-order Wronskian W

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -Py_1' - Qy_1 & -Py_2' - Qy_2 & -Py_3' - Qy_3 \end{vmatrix} = 0$$

vanishes because the 3d row is a linear combination of the first two rows. So the three solutions must be linearly dependent.