Review of Separation of Variables

The Helmholtz equation in cartesian coordinates:

$$0 = (\Delta + k^2)\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

We let

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

with

$$X'' = -k_x^2 X, \quad Y'' = -k_y^2 Y, \quad Z'' = -k_z^2 Z$$

and

$$\vec{k}^2 = +h^2$$

Example

$$X = \cos(k_x x)$$
$$Y = \sin(k_y y)$$
$$Z = \cos(k_z z)$$

with

$$k_x^2 + k_y^2 + k_z^2 = +k^2,$$

Pages 106-139 contain my notes on the separation of variables.

Singular Points

Consider the 2d-order ODE

$$y'' = f(x, y, y').$$

Suppose $y'' = f(x_0, y, y')$ is finite for all finite y and y'. Then x_0 is a regular point of the ODE.

But if $y'' = f(x_0, y, y')$ diverges for any pair of finite values (y, y'), then x_0 is a singular point of the ODE.

If the ODE is of the form

$$y'' + P(x)\, y' + Q(x)\, y = 0,$$

and both $P(x_0)$ and $Q(x_0)$ are finite, then x_0 is a regular point of the ODE. But if $P(x_0)$ or $Q(x_0)$ or both are infinite, then x_0 is a singular point.

Not all singular points are equally bad.

If $P(x)$ or $Q(x)$ diverges as $x \to x_0$, but

$(x-x_0)\, P(x)$ and $(x-x_0)^2\, Q(x)$ remain finite as $x \to x_0$, then x_0 is a <u>regular</u> singular point.

Regular singular points are also called <u>nonessential</u> singular points.

But if either $(x-x_0)P(x)$ or $(x-x_0)^2Q(x)$ diverges as $x \to x_0$, then x_0 is an <u>irregular</u> singular point or an <u>essential</u> singularity.

To analyze the point at infinity, we set

$$z = \frac{1}{x} \qquad \text{and look at } z = 0.$$

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dx} = \frac{dz}{dx}\frac{dy(z^{-1})}{dz} = \frac{dz}{dx}\dot{y}$$

$$= -\frac{1}{x^2}\dot{y} = -z^2\dot{y}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}\frac{dy'}{dz} = -z^2\frac{d}{dz}\left(-z^2\dot{y}\right)$$

$$= z^4\ddot{y} + 2z^3\dot{y}, \qquad \text{so}$$

$$0 = y'' + P(z^{-1})y' + Q(z^{-1})y \qquad \text{becomes}$$

$$0 = z^4\ddot{y} + \left[2z^3 - z^2P(z^{-1})\right]\dot{y} + Q(z^{-1})y \qquad \text{or}$$

$$0 = \ddot{y} + \left(\frac{2z - P}{z^2}\right)\dot{y} + \frac{Q}{z^4}y.$$

If

$$\frac{2z - P(z^{-1})}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4}$$

remain finite as $z \to 0$, then $x_0 = \infty$ is a regular point of the ODE.

If

$$\frac{2z - P(z^{-1})}{z} \quad \text{and} \quad \frac{Q(z^{-1})}{z^2}$$

remain finite as $z \to 0$, then $x_0 = \infty$ is a regular singular point. Otherwise $x_0 < \infty$ is an irregular singular point or an essential singularity.

Problem 8.4.1 Legendre's equation is

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0 \qquad \text{or}$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{\ell(\ell+1)}{1-x^2}y = 0$$

Clearly $x = \pm 1$ are singular points. But

$$-(x-1)\frac{2x}{1-x^2} = \frac{2x}{1+x} \to 1 \quad \text{as } x \to 1$$

and

$$(x-1)^2\frac{1}{(1-x^2)} = \frac{x-1}{(x+1)} \to 0 \quad \text{as } x \to 1$$

So $x_0 = 1$ is regular.

$$-(x+1)\frac{2x}{1-x^2} = \frac{2x}{1-x} \to -1 \quad \text{as} \quad x \to -1$$

$$\frac{(x+1)^2}{1-x^2} = \frac{x+1}{1-x} \to 0 \quad \text{as} \quad x \to -1$$

So $x_0 = -1$ is regular.

$$\left(2z + \frac{2z^{-1}}{1-z^{-2}}\right)\frac{1}{z^2} = \frac{2}{z} + \frac{2}{z^3 - z}$$

diverges as $z \to 0$, so $x_0 = \infty$ is a singular point. But

$$2 + \frac{2}{z^2-1} \to 0 \quad \text{as} \quad z \to 0 \qquad \text{and}$$

$$\left(\frac{1}{1-z^{-2}}\right)\frac{1}{z^2} = \frac{1}{z^2-1} \to -1 \quad \text{as } z \to 0$$

So $x_0 = \infty$ is a regular singular point, like $x_0 = \pm 1$.

Series Solutions — Frobenius's Method

Consider the general linear, 2d-order, homogeneous ODE

$$y'' + P y' + Q y = 0.$$

We try
$$y(x) = x^k \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda \qquad a_0 \neq 0.$$

Since $P(x)$ and $Q(x)$ contribute powers of x, this gets messy fast if we try to stay general. So we look at

$$y'' + \omega^2 y = 0$$

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$y' = \sum_{\lambda=0}^{\infty} (k+\lambda) a_\lambda x^{k+\lambda-1}$$

$$y'' = \sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2}$$

So

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + \omega^2 \sum a_\lambda x^{k+\lambda} = 0$$

The most singular term at $x=0$ is

$$a_0 k(k-1) x^{k-2}$$

and so $\qquad k(k-1) = 0.$

This is called the indicial equation

So $\quad k=0 \quad$ or $\quad k=1.$

Let $\quad j=\lambda-2.$ Then the ODE is

$$\sum_{j=-2}^{\infty} a_{j+2}(k+j+2)(k+j+1)\, x^{k+j} + \omega^2 \sum_{j=0}^{\infty} a_j \, x^{k+j} = 0$$

So

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0 \qquad \text{or}$$

$$a_{j+2} = -\frac{\omega^2}{(k+j+2)(k+j+1)}\, a_j \quad ,$$

which is a two-term recurrence relation.

Case $k=0$: Then the worst term for $j=-2$

$$a_0 \,(0)(-1)(-1)\, x^{-2} = 0$$

and the next-worst term is for $j=-1$

$$a_1 \,(0-1+2)(0-1+1)\, x^{-1} = 0$$

So a_1 is arbitrary. We set $a_1=0$, whence

$$0 = a_3 = a_5 = a_7 \quad \cdots \quad a_{2n+1} = 0 .$$

Then for $j = 0$ and $k = 0$ $\quad a_{j+2} = -\dfrac{\omega^2}{(j+2)(j+1)} a_j$

and so

$$a_2 = -\frac{\omega^2}{2 \cdot 1} a_0 = -\frac{\omega^2}{2} a_0$$

$j = 2$

$$a_4 = -\frac{\omega^2}{4 \cdot 3} a_2 = \frac{\omega^4}{4!} a_0$$

$j = 4$

$$a_6 = -\frac{\omega^2}{6 \cdot 5} a_4 = -\frac{\omega^6}{6!} a_0$$

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0$$

$$y(x) = a_0 \sum (-1)^n \frac{\omega^{2n}}{(2n)!} = a_0 \cos \omega x .$$

Case $k = 1$:

$$a_{j+2} = -\frac{\omega^2}{(j+3)(j+2)} a_j$$

The worst term is

$$a_0 (1-2+2)(1-2+1) x^{-1} = 0 .$$

The other problematic term

$a_1 (1-1+2)(1-1+1) x^0$ must vanish.

So we must set $a_1 = 0$ if $k = 1$.
So

$$a_{2n+1} = 0 \quad \text{again.}$$

And

$j=0$ $$a_2 = -\frac{\omega^2}{3\cdot 2} a_0 = -\frac{\omega^2}{3!} a_0$$

$j=2$ $$a_4 = -\frac{\omega^2}{5\cdot 4} a_2 = \frac{\omega^4}{5!} a_0$$

$j=4$ $$a_6 = -\frac{\omega^2}{7\cdot 6} a_4 = -\frac{\omega^6}{7!} a_0 \quad \text{etc.}$$

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

So for $k=1$

$$y(x) = \sum_{n=0}^{\infty} a_{2n} x^{1+2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n}}{(2n+1)!} x^{2n+1} a_0$$

$$= \frac{a_0}{\omega} \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!}$$

$$= \frac{a_0}{\omega} \sin \omega x.$$

So we got two independent solutions for the ODE

$$y'' + \omega^2 y = 0.$$

Some ODE's require us to work harder for two solutions.

Sometimes one expands about another point

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{k+\lambda} \qquad a_0 \neq 0.$$

Suppose we write the ODE as

$$0 = \mathcal{L}(x)\, y(x) = y''(x) + P(x)\, y'(x) + Q(x)\, y(x),$$

by which we mean

$$\mathcal{L}(x) = \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x).$$

Then

$$\mathcal{L}(-x) = \frac{d^2}{dx^2} - P(-x)\frac{d}{dx} + Q(-x).$$

If $\mathcal{L}(-x) = \pm\mathcal{L}(x)$, then

$\mathcal{L}(x)\, y(x) = 0$ implies $\mathcal{L}(-x)\, y(-x) = 0$

and $\mathcal{L}(-x) = \pm \mathcal{L}(x)$ further implies

$$\pm \mathcal{L}(x)\, y(-x) = 0 \quad \text{or}$$

$$\mathcal{L}(x)\, y(-x) = 0.$$

In this case, both $y(x)$ and $y(-x)$ is a solution of

$$\mathcal{L}(x)\, y(x) = 0$$

and we may resolve $y(x)$ into even and odd solutions

$$y(x) = \tfrac{1}{2}\left[y(x) + y(-x)\right] + \tfrac{1}{2}\left[y(x) - y(-x)\right]$$

$$= y_e(x) + y_o(x).$$

$$\mathcal{L}_L(x) = \frac{d^2}{dx^2} - \frac{2x}{1-x^2}\frac{d}{dx} + \frac{\ell(\ell+1)}{1-x^2} = \mathcal{L}_L(-x)$$

$$\mathcal{L}_C(x) = \frac{d^2}{dx^2} - \frac{x}{1-x^2} + \frac{n^2}{1-x^2} = \mathcal{L}_C(-x)$$

$$\mathcal{L}_B(x) = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{x^2-n^2}{x^2} = \mathcal{L}_B(-x)$$

$$\mathcal{L}_H(x) = \frac{d^2}{dx^2} + \omega^2 = \mathcal{L}_H(-x)$$

These are all even operators.

What can go wrong? Try Bessel's equation

$$0 = y'' + \frac{1}{x} y' + \left(\frac{x^2-n^2}{x^2}\right) y = x^2 y'' + x y' + (x^2 - n^2) y$$

Let
$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$0 = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda}$$
$$+ \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda} \qquad \text{(BE)}$$

Set $\lambda = 0$ to isolate the terms with x^k:

$$a_0 [k(k-1) + k - n^2] = 0$$

So $k^2 = n^2$ is the indicial equation.

The x^{k+1} terms are

$$a_1 [(k+1)k + k+1 - n^2] = a_1 (k+1+n)(k+1-n)$$

Now if $k = \pm n$, these terms don't vanish unless $k = n = -1/2$ or $k = -n = 1/2$. In all other cases, we set $a_1 = 0$.

Set $k = n$. The terms with x^{n+j} in (BE) are

$$a_j [(n+j)(n+j-1) + n+j - n^2] + a_{j-2} = 0$$

$$a_j [j(j-1) + nj + n(j-1) + n + j] + a_{j-2} = 0$$

or $$a_j\left[j^2+2nj\right] = -a_{j-2}$$

$$a_j\, j\,(j+2n) = -a_{j-2}$$ or

$$a_{j+2} = -\frac{a_j}{(j+2)(j+2+2n)}$$

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0\, n!}{2^2\, 1!\,(n+1)!}$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{4\cdot 2\,(2n+4)(2n+2)}$$

$$= \frac{a_0\, n!}{2^4\, 2!\,(n+2)!}$$

$$a_6 = -\frac{a_4}{6(2n+6)} = -\frac{a_0\, n!}{2^6\, 3!\,(n+3)!}$$

So

$$a_{2p} = (-1)^p \frac{a_0\, n!}{2^{2p}\, p!\,(n+p)!}$$ and

$$y(x) = a_0 \sum_{j=0}^{\infty} (-1)^j \frac{n!\, x^{n+2j}}{2^{2j}\, j!\,(n+j)!}$$

$$y(x) = a_0 \, 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j! \, (n+j)!} \left(\frac{x}{2}\right)^{n+2j}$$

$$= a_0 2^n n! \; J_n(x) \qquad (11.5).$$

Note that

$$J_n(-x) = (-1)^n \, J_n(x)$$

If $k = -n$ and n is not an integer, then we may generate a second solution $J_{-n}(x)$. Now

$$a_j \left[(j-n)(j-1-n) - n + j - n^2 \right] + a_{j-2} = 0$$

$$a_j \left[j(j-1) - n(j-1) - nj - n + j \right] + a_{j-2} = 0$$

$$a_j \left(j^2 - 2nj \right) + a_{j-2} = 0$$

$$a_j \, j \, (j - 2n) + a_{j-2} = 0$$

$$a_{j+2} = - \frac{a_j}{(j+2)(j+2-2n)}$$

Now if $n \geq 0$ is a positive integer and j is an even positive integer, there can be trouble when

$$j + 2 = 2n.$$

In this case, one sets

$$J_{-n}(x) = (-1)^n J_n(x)$$

and we do not get a second solution.

But what if we expand about a singular point? Consider

$$x^2 y'' = 6y \quad \text{and try} \quad y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

$$\sum a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} = 6 \sum a_\lambda x^{k+\lambda}$$

or

$$\sum_{\lambda=0}^{\infty} a_\lambda \left[(k+\lambda)(k+\lambda-1) - 6 \right] x^{k+\lambda} = 0$$

We now need $(k+\lambda)(k+\lambda-1) = 6$ for all λ

Impossible. Try for $\lambda = 0$

$$k(k-1) = 6 \qquad 0 = k^2 - k - 6$$

$$k = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} = 3 \ \& \ -2.$$

So we get two solutions

$$y = x^3 \quad \text{and} \quad y = x^{-2}.$$

Here $x = 0$ is a regular singular point of $0 = y'' - \frac{6}{x^2} y$.