So the Fourier transform $\widetilde{f*g}(\omega)$ of the convolution $f*g(x)$ of two functions $f(x)$ and $g(x)$ is the product of their Fourier transforms

$$\widetilde{f*g}(\omega) = \hat{f}(\omega)\,\hat{g}(\omega).$$

Convolutions are common in physics. The electrostatic potential $\phi(\vec{x})$ is the convolution of the charge density $\rho(\vec{x}')$

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

with $(2\pi)^{3/2}/|\vec{x}|$. Similarly, the vector potential of magnetostatics $\vec{A}(\vec{x})$ is the convolution

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

of the current density $\vec{J}(\vec{x})$ with $(2\pi)^{3/2}/|\vec{x}|$.

These convolutions are three dimensional. So to see why these convolutions occur in physics, we must understand multi-dimensional Fourier transforms.

The Fourier transform $\tilde{f}(\vec{k})$ of a function $f(\vec{x})$ of 3 variables is

$$\tilde{f}(\vec{k}) = \int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{2\pi}} e^{ik_1x_1} \int_{-\infty}^{\infty} \frac{dx_2}{\sqrt{2\pi}} e^{ik_2x_2} \int_{-\infty}^{\infty} \frac{dx_3}{\sqrt{2\pi}} e^{ik_3x_3} f(\vec{x})$$

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{k} = (k_1, k_2, k_3)$. So

$$\tilde{f}(\vec{k}) = \int_{-\infty}^{\infty} \frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} f(\vec{x}).$$

Now since

$$\delta^3(\vec{x}-\vec{x}') = \delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} e^{\pm ik_1(x_1-x_1')} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{\pm ik_2(x_2-x_2')} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} e^{\pm ik_3(x_3-x_3')}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}')},$$ we see that

$$\int e^{-i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k}) \frac{d^3k}{(2\pi)^{3/2}} = \int \frac{d^3k\, d^3x'}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})} f(\vec{x}')$$

$$= \int d^3x'\, \delta(\vec{x}-x')\, f(\vec{x}') = f(\vec{x}).$$

So the completeness relations

$$\delta^{(3)}(\vec{x}-\vec{x}\,') = \int \frac{d^3k}{(2\pi)^3}\, e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

and

$$\delta^{(3)}(\vec{k}-\vec{k}\,') = \int \frac{d^3x}{(2\pi)^3}\, e^{\pm i\vec{x}\cdot(\vec{k}-\vec{k}\,')}$$

imply the reciprocal transforms

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{f}(\vec{k}) \quad \text{and}$$

$$\hat{f}(\vec{k}) = \int \frac{d^3x}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\, f(\vec{x}).$$

In n dimensions,

$$\delta^{(n)}(\vec{x}-\vec{x}\,') = \int \frac{d^nk}{(2\pi)^n}\, e^{\pm i\vec{k}\cdot(\vec{x}-\vec{x}\,')},$$

$$f(\vec{x}) = \int \frac{d^nk}{(2\pi)^{n/2}}\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{f}(\vec{k}),$$

and

$$\hat{f}(\vec{k}) = \int \frac{d^nx}{(2\pi)^{n/2}}\, e^{i\vec{k}\cdot\vec{x}}\, f(\vec{x})$$

where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{k} = (k_1, \ldots, k_n)$.

In the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, the magnetostatic vector potential satisfies

$$\Delta \vec{A}(\vec{x}) = \nabla^2 \vec{A}(\vec{x}) = -\frac{4\pi}{c}\vec{J}(\vec{x}).$$

Let

$$\vec{A}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{A}(\vec{k})\, e^{-ik\cdot x} \quad \text{and}$$

$$\vec{J}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{J}(\vec{k})\, e^{-ik\cdot x}.$$

Then

$$\nabla^2 \vec{A}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{A}(\vec{k})\, \nabla^2 e^{-ik\cdot x}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{A}(\vec{k})\,(-\vec{k}^2)\, e^{-ik\cdot x}$$

$$= -\frac{4\pi}{c}\vec{J}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{J}(\vec{k})\left(-\frac{4\pi}{c}\right) e^{-ik\cdot x}.$$

So

$$-\vec{k}^2 \tilde{A}(\vec{k}) = -\frac{4\pi}{c}\tilde{J}(\vec{k}) \quad \text{or}$$

$$\tilde{A}(k) = \frac{1}{c}\frac{4\pi}{\vec{k}^2}\tilde{J}(\vec{k}).$$

But if the Fourier transform $\hat{A}(\vec{k})$ is the product of $\hat{\vec{J}}(\vec{k})$, the Fourier transform of the current density $\vec{J}(\vec{x})$, and $\frac{4\pi}{\vec{k}^2}$ is the Fourier transform of something else. What? Well, the something else, $G(\vec{x})$, must be

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{4\pi}{\vec{k}^2} e^{-i\vec{k}\cdot\vec{x}}$$

$$= \frac{8\pi^2}{(2\pi)^{3/2}} \int_{-1}^{1} d\mu \int_0^\infty dk \; e^{-ikr\mu} \qquad r = |\vec{x}|$$

$$= 2\sqrt{2\pi} \int_0^\infty dk \left(\frac{e^{-ikr} - e^{ikr}}{-ikr} \right)$$

$$= \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dk}{k} \frac{e^{ikr} - e^{-ikr}}{2i} = \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dk}{x} \sin kr$$

$$= \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dx}{x} \sin x = 4\sqrt{2\pi} \, \frac{\pi}{2} \, \frac{1}{r} = \frac{(2\pi)^{3/2}}{r}$$

by exercise 7.2.15.

So $G(\vec{x}) = \dfrac{(2\pi)^{3/2}}{|\vec{x}|}$ and

$$\tilde{\vec{A}}(k) = \frac{1}{c}\,\tilde{G}(\vec{k})\,\tilde{\vec{J}}(\vec{k})$$

whence

$$\vec{A}(x) = \frac{1}{c}\frac{1}{(2\pi)^{3/2}}\int d^3x'\, G(x-x')\,\vec{J}(x')$$

$$= \frac{1}{c}\frac{1}{(2\pi)^{3/2}}\int d^3x'\,\frac{(2\pi)^{3/2}}{|\vec{x}-\vec{x}'|}\,\vec{J}(\vec{x}')$$

$$= \frac{1}{c}\int \frac{d^3x'}{|\vec{x}-\vec{x}'|}\,\vec{J}(\vec{x}').$$

The 3-d convolution theorem is

$$f * g\,(x) = \frac{1}{(2\pi)^{3/2}}\int d^3y\; f(\vec{x}-\vec{y})\,g(\vec{y})$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^3y\; g(\vec{x}-\vec{y})\,f(\vec{y}) \quad \text{and}$$

$$\widetilde{f * g}(\vec{k}) = \tilde{f}(\vec{k})\,\tilde{g}(\vec{k}).$$

Going back to the differential equation

$$\nabla^2 \vec{A}(\vec{x}') = -\frac{4\pi}{c} \vec{J}(\vec{x}'),$$

we see that the relation

$$\nabla^2 \vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \; \nabla^2 \frac{1}{|x-x'|} \; \vec{J}(x')$$

$$= -\frac{4\pi}{c} \vec{J}(\vec{x})$$

implies that

$$-\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \; \delta^{(3)}(\vec{x}-\vec{x}').$$

So

$$G(\vec{x},\vec{x}') = \frac{1}{4\pi |\vec{x}-\vec{x}'|}$$

is the Green's function for $-\nabla^2$:

$$-\nabla^2 \frac{1}{4\pi |\vec{x}-x'|} = \delta^{(3)}(\vec{x}-\vec{x}').$$

↓

Yet another representation of Dirac's delta function!

If $$-\nabla^2 G(\vec{x}, \vec{x}') = \delta^{(3)}(\vec{x} - \vec{x}'),$$ then

$$G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{k}^2}$$

for then

$$-\nabla^2 G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta^{(3)}(\vec{x} - \vec{x}').$$

And

$$G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2} = \frac{1}{4\pi|\vec{x} - \vec{x}'|}$$

according to the integration on p. 341.

So

$$-\nabla^2 \frac{1}{4\pi|\vec{x} - \vec{x}'|} = \delta^{(3)}(\vec{x} - \vec{x}').$$

Note

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi|\vec{x} - \vec{x}'|}$$

is the Green's function, while on p. 341,

$$G(\vec{x}) = \frac{(2\pi)^{3/2}}{r}.$$

This delta-function helps us solve

$$\nabla \cdot E = 4\pi \rho .$$

First, in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$, the divergence of

$$\vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad \text{is} \quad \nabla \cdot E = -\nabla^2 \phi$$

and so

$$-\nabla^2 \Phi = 4\pi\rho .$$

We now form the convolution

$$\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

and note that

$$-\nabla^2 \Phi(x) = \int \rho(\vec{x}') \left(-\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'$$

$$= 4\pi \int \rho(\vec{x}') \, \delta^{(3)}(\vec{x} - \vec{x}') \, d^3x'$$

$$= 4\pi \, \rho(\vec{x}).$$

So in the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$,

$$\Phi(\vec{x}, t) = \int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} d^3x'$$

even in the time-dependent case,

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

and

$$\nabla \cdot E = -\nabla^2 \phi = 4\pi\rho.$$

In the static case, $\vec{A}(\vec{x})$ is given by

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \quad , \qquad \text{but}$$

what about the time-dependent $\vec{A}(\vec{x},t)$?
We start with $B = \nabla \times A$ and

$$\overrightarrow{\nabla \times B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\,\partial_{k\ell m}\,\partial_j\partial_\ell A_m = \frac{4\pi}{c}J_i + \frac{1}{c}\frac{\partial E_i}{\partial t}$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\partial_j\partial_\ell A_m = -\partial_j^2 A_i \qquad \text{since } \nabla\cdot A = 0$$

so

$$-\partial_j^2 A_i = \frac{4\pi}{c}J_i + \frac{1}{c}\frac{\partial}{\partial t}\left(-\partial_i\phi - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) \qquad \text{or}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)A_i = \frac{4\pi}{c}\left(J_i - \frac{1}{4\pi}\partial_i\dot{\phi}\right) \equiv \frac{4\pi}{c}\vec{J}_t$$

where $\nabla\cdot J_t = \nabla\cdot J - \frac{1}{4\pi}\nabla^2\dot{\phi} = \nabla\cdot J + \dot{\rho} = 0$,
$\vec{J}_t$ is the transverse current density.
The equation $0 = \vec{\nabla}\cdot\vec{J} + \dot{\rho}$ is called current conservation.

So

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) A_i = \frac{4\pi}{c} J_{t\,i} \qquad \text{or}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \vec{A} = \Box \vec{A} = \frac{4\pi}{c} \vec{J}_t .$$

We seek a Green's function for $\Box$:

$$\left(\frac{1}{c^2}\frac{\partial^3}{\partial t^2} - \Delta\right) G(x,t;x',t') = 4\pi\, \delta^3(x-x')\, \delta(t-t')$$

So we write

$$G(\vec{x},t;\vec{x}',t') = \int d^3k \int_{-\infty}^{\infty} d\omega\, e^{ik(x-x') - i\omega(t-t')}\, g(\vec{k},\omega)$$

and get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = \int d^3k \int d\omega \left(\vec{k}^2 - \frac{\omega^2}{c^2}\right) e^{i\vec{k}\cdot(\vec{x}-\vec{x}') - i\omega(t-t')}\, g(\vec{k},\omega)$$

$$= 4\pi \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}') - i\omega(t-t')}$$

So

$$g(\vec{k},\omega) = \frac{1}{4\pi^3} \frac{1}{\vec{k}^2 - \frac{\omega^2}{c^2}} ,$$

but we need to interpret the singularity at

$$k^2 = \frac{\omega^2}{c^2} .$$

Our formula for $A(\vec{x}, t)$ in the Coulomb gauge is

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\int d^3x'\,dt'\; G(\vec{x}, t; \vec{x}', t')\, \vec{J}_t(x', t').$$

So we want G to vanish when $t' > t$. So we put the poles in the LHP:

$$G(\vec{x}, t; \vec{x}', t') = \int d^3k \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\vec{k}\cdot\vec{r} - i\omega\tau}}{4\pi^3}\, \frac{1}{(k^2 - \frac{1}{c^2}(\omega + i\epsilon))^2}$$

where $\vec{r} = \vec{x} - \vec{x}'$ and $\tau = t - t'$. For $\tau < 0$,

$$\int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega\tau}}{k^2 - \frac{1}{c^2}(\omega + i\epsilon)^2} = \int \frac{dz\, e^{+i|\tau| z}}{k^2 - \frac{1}{c^2}(z + i\epsilon)^2} = 0.$$

So $G(r, \tau) = 0$ for $\tau < 0$. For $\tau > 0$,

$$\int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega\tau}}{k^2 - \frac{1}{c^2}(\omega + i\epsilon)^2} = \int \frac{dz\, e^{-iz\tau}}{k^2 - \frac{1}{c^2}(z + i\epsilon)^2}$$

$$= -c^2 \oint dz \frac{e^{-iz\tau}}{(z + i\epsilon - kc)(z + i\epsilon + kc)}$$

$$= + c^2 2\pi i \left[\frac{e^{-i\tau(-kc)}}{-2kc} + \frac{e^{-i\tau kc}}{2kc} \right]$$

$$= \frac{2\pi}{c} \frac{1}{k} \left(\frac{e^{i\tau kc} - e^{-i\tau kc}}{2i} \right) .$$ So

$$G(\vec{r},\tau) = \frac{c}{2\pi^2} \int d^3k \frac{1}{k} \sin(kc\tau)\, e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{c}{\pi} \int_{-1}^{1} d\mu \int_0^\infty dk \; k \sin(kc\tau)\, e^{ikr\mu}$$

$$= \frac{c}{\pi} \int_0^\infty dk \; k \sin(kc\tau) \frac{e^{ikr} - e^{-ikr}}{ikr}$$

$$= \frac{2c}{\pi r} \int_0^\infty dk \; \sin(kr) \sin(ck\tau) .$$

Let $x = ck$. The integrand is even in x

$$G(\vec{r},\tau) = \frac{1}{\pi r} \int_{-\infty}^{\infty} dx \left(\frac{e^{i\frac{xr}{c}} - e^{-i\frac{xr}{c}}}{2i} \right) \left(\frac{e^{ix\tau} - e^{-ix\tau}}{2i} \right)$$

$$= \frac{1}{4\pi r} \int_{-\infty}^{\infty} dx \left[e^{i(\tau - \frac{r}{c})x} - e^{i(\tau + \frac{r}{c})x} + e^{i(\frac{x}{c} - \tau)x} - e^{-i(\tau + \frac{r}{c})x} \right]$$

which is just a sum of

delta functions:

$$G(\vec{r},\tau) = \frac{1}{r}\left[\delta(\tau - \tfrac{r}{c}) - \delta(\tau + \tfrac{r}{c})\right].$$

But here $\tau > 0$, so

$$G(\vec{r},\tau) = \frac{\delta(\tau - r/c)}{r}$$

or

$$G(\vec{x},t;\vec{x}',t') = \frac{\delta(t - t' - |\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}.$$

This retarded Green's function satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\vec{x},t;\vec{x}',t') = 4\pi\delta^3(\vec{x}-\vec{x}')\,\delta(t-t').$$

So in the Coulomb gauge, $\nabla\cdot A = 0$, the time-dependent vector potential $\vec{A}(\vec{x},t)$ is

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x'\,dt'\,\frac{\delta(t-t'-|x-x'|/c)}{|\vec{x}-\vec{x}'|}\,\vec{J}_t(x',t')$$

and it satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A}(\vec{x},t) = \frac{4\pi}{c}\,\vec{J}_t(x,t).$$

If we do the t' integral, then the vector potential

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \, \frac{\vec{J}_t(x', t - |\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}$$

depends upon the current density J at the earlier time

$$t' = t - \frac{|\vec{x}-\vec{x}'|}{c} .$$

But, as we saw on page 345, the scalar potential in the Coulomb gauge, $\nabla\cdot A = 0$, is "instantaneous":

$$\Phi(\vec{x},t) = \int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\, d^3x'.$$

The Coulomb-gauge condition

$$\vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0$$

is not Lorentz invariant, so electrodynamics in the Coulomb gauge looks as though it violates relativity. But it doesn't. Electrodynamics in the Coulomb gauge does respect special relativity.

The Coulomb gauge is also called the radiation gauge.

The Lorentz gauge condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = \partial_\mu A^\mu = 0$$

is itself Lorentz invariant.

Why are convolutions ubiquitous? Because space-time is homogeneous. The convolution

$$A(x) = \int d^4x' \; J(x') \; f(x-x')$$

respects the homogeneity of space-time. Suppose

$$A_2(x) = \int d^4x' \; J_2(x') f(x-x')$$

$$A_1(x) = \int d^4x' \; J_1(x') \; f(x-x') \qquad \text{and}$$

$$J_2(x) = J_1(x+b)$$

ie., the two sources differ by a translation b.
Then

$$A_2(x) = \int d^4x' \; J_1(x'+b) \; f(x-x')$$

$$= \int d^4x' \; J_1(x') \; f(x-(x'-b))$$

$$= \int d^4x' \; J_1(x') \; f(x+b-x') = A_1(x+b)$$

ie., the fields differ only by the same translation b. So the symmetry that spacetime is homogeneous leads to convolutions. The dynamics might be

$$\mathcal{D}(-i\nabla) \; A(x) = J(x),$$

Let $$f(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{\mathcal{D}(k)}$$ so that

$$\mathcal{D}(-i\nabla) f(x-x') = \delta^{(4)}(x-x').$$