Cauchy's inequality:

Suppose $f(z)$ is analytic in a region that includes the circle

$$z = re^{i\theta},$$

and suppose that on this circle $f(z)$ is bounded by

$$|f(z)| \leq M .$$

Then since (6.47)

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{z^{n+1}} ,$$ we have

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi} \oint \frac{|f(z)|\,|dz|}{|z|^{n+1}}$$

$$= \frac{n!}{2\pi} M \int_0^{2\pi} \frac{r\,d\theta}{r^{n+1}} = \frac{n!}{r^n} M .$$

Now suppose that $f(z)$ is analytic everywhere and bounded by

$$|f(z)| \leq M .$$

Then

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} M \rightarrow 0 \text{ as } r \rightarrow \infty.$$

So every derivative at $r=0$ vanishes.

$$f^{(n)}(0) = 0 \qquad n \geq 1.$$

So $f(z)$ which has the power series expansion (6.57)

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^{(n)}(0)$$

$$= f^{(0)}(0) = f(0)$$

$f(z)$ is a constant. This is Liouville's theorem.

G. F. Gauss applied it to the case

$$f(z) = \frac{1}{P(z)} = \frac{1}{\sum_{m=0}^{n} a_m z^m}$$

of an inverse polynomial. Suppose the equation

$$P(z) = 0$$

has no solution over the complex plane. Then

$$P(z) \neq 0.$$

Also $\qquad P(z) \sim a_n z^n$ for large z.

So

$$|f(z)| = \frac{1}{P(z)}$$

is then bounded and analytic for all z. By L's thm., it must be constant.

But $f(z) = P(z)^{-1}$ is not constant.

So the equation $P(z) = \sum_{m=0}^{n} a_m z^m = 0$

must have at least one solution.

Say $P(z_1) = 0$. Then

$$P(z) = (z - z_1) \sum_{m=0}^{n-1} b_m z^m = (z - z_1) P_1(z).$$

But $P_1(z)$ must also have a root. Hence

$$P(z) = \sum_{m=0}^{n} a_m z^m \quad \text{has } n \text{ roots.}$$

$$\sum_{m=0}^{n} a_m z^m = a \prod_{i=1}^{n} (z - z_i)$$

This is the fundamental theorem of algebra: every nth-order polynomial has n complex roots.

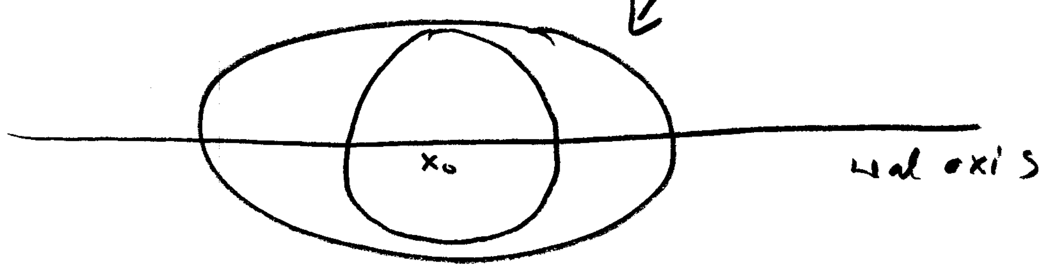
They may coincide, for instance,

$$P(z) = z^2 + 2z + 1 = (z+1)^2$$

has the root $z = -1$ twice.

Schwarz reflection principle.

If $f(z)$ is analytic here



Then

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - x_0)^n}{n!} f^{(n)}(x_0)$$

within a disc around x_0, real

Say $f(z) = f(z)^*$ when $z = x$ is real.

Then
$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

is real and we find that $f^n(x_0)$ is real for all n. In this case then
$$f^*(z) = \left[\sum_{n=0}^{\infty} \frac{(z-x_0)^n}{n!} f^{(n)}(x_0) \right]^*$$

$$= \sum_{n=0}^{\infty} \frac{(z^*-x_0)^n}{n!} f^{(n)}(x_0) = f(z^*),$$

So
$$f^*(z) = f(z^*).$$

Analytic Continuation

$$f(z) = \frac{1}{1+z}$$ has a pole at $z = -1$.

and is otherwise analytic.

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$$ converges for $|z| < 1$.

Now expand $f(z)$ about $z = i$:

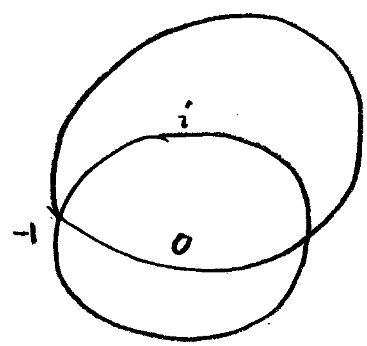
$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+z-i} = \frac{1}{(1+i)\left[1 + \frac{z-i}{1+i}\right]}$$

$$= \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{i-z}{1+i}\right)^n$$

a series that converges for $\left|\frac{i-z}{1+i}\right| < 1$ ie

for

$$|z-i| < |1+i| = \sqrt{2}$$



So a power series about $z=0$ defines $f(z)$ about $z=i$ and a power series about $z=i$ defines $f(z)$ out to $z=(1+\sqrt{2})i$. This is called analytic continuation.

Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $|x| < R$. Then this series defines an analytic function there

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for } |z| < R.$$

This is called the "permanence of the algebraic form.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

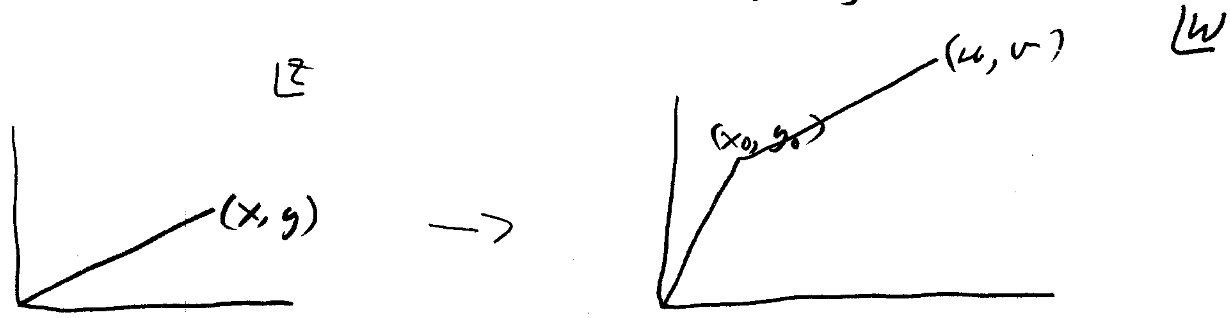
$$= \frac{e^{iz} - e^{-iz}}{2i}.$$

Complex functions $w = f(z)$ map the complex z-plane into the complex w-plane.

Translation: $\quad W = z + z_0 \quad$ for $\quad f(z) = z + z_0$

If $z_0 = x_0 + iy_0$, and $z = x + iy$, and $w = u + iv$ then

$$u = x_0 + x \qquad v = y_0 + y$$



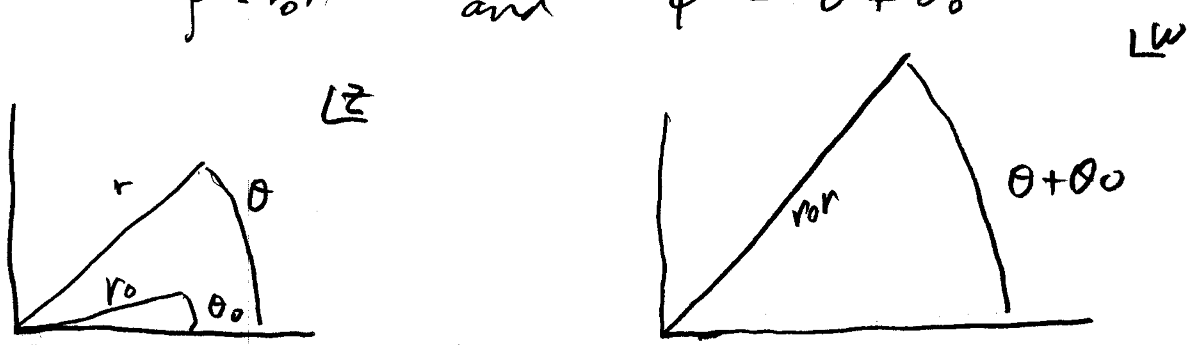
Rotation: $f(z) = z_0 z$, so $w = z_0 z$

Let $w = \rho e^{i\varphi}$, $\quad z = re^{i\theta}$, and $z_0 = r_0 e^{i\theta_0}$.
Then

$$\rho e^{i\varphi} = r_0 e^{i\theta_0} r e^{i\theta} = r_0 r e^{i(\theta + \theta_0)}$$

so

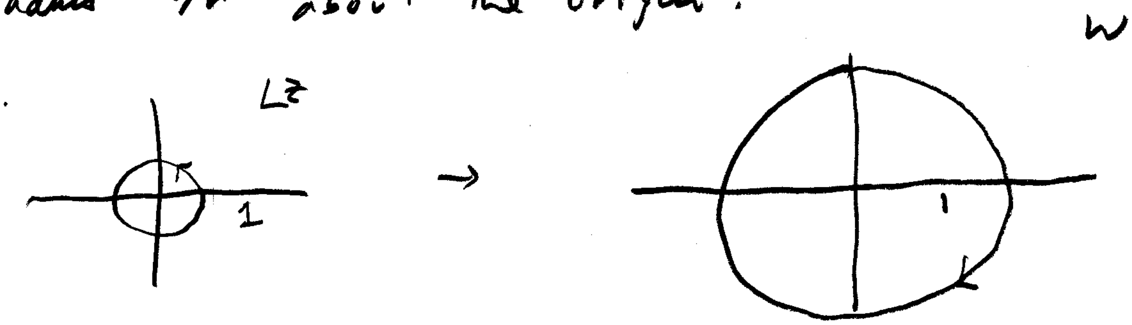
$$\rho = r_0 r \quad \text{and} \quad \varphi = \theta + \theta_0$$



Inversion: $\quad u + iv = \dfrac{1}{x+iy}$

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

So a circle of radius r in the z-plane about the origin is mapped into a circle of radius $1/r$ about the origin:



But a straight line can map into a circle!

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \qquad \text{and}$$

$$x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2} \qquad \text{so the line } y = c_1$$

maps into

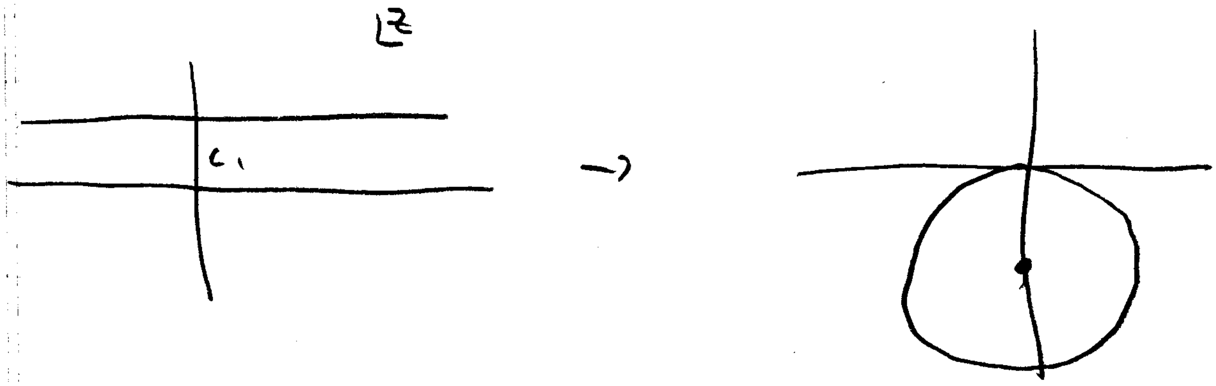
$$c_1 = \frac{-v}{u^2+v^2}$$

or $\quad u^2 + v_2 + \dfrac{v}{c_1} = 0$

$$u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \frac{1}{(2c_1)^2}$$

which is a circle in the w-plane of radius $1/2|c_1|$.

centred at $u = 0$, $v = -\frac{1}{2c_1}$.



Inversion maps
circles & straight lines into circles
and straight lines.

Branch Points, Cuts, Etc.

Suppose $w = \rho e^{i\varphi} = f(z) = z^2 = r^2 e^{2i\theta}$.

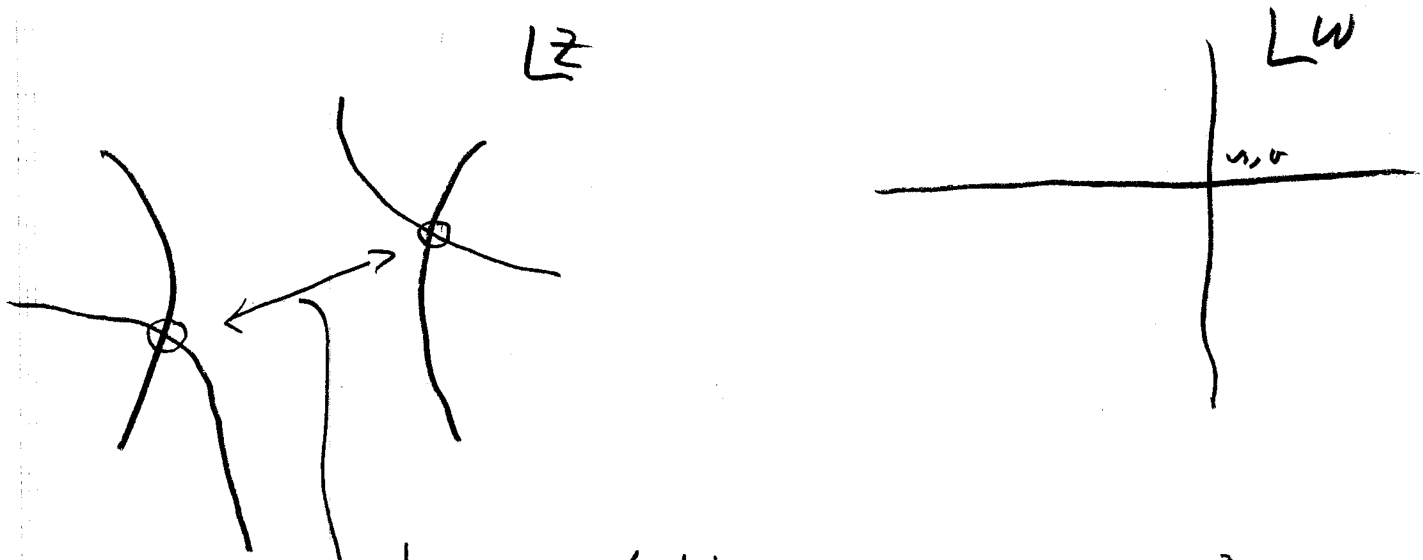
$\rho = r^2$ and $\varphi = 2\theta$.

So the UH z-plane maps onto the whole w-plane, and the LHZP also maps onto the whole w-plane.

$u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$

So (u,v) corresponds to the two hyperbolae

$u = x^2 - y^2 \qquad v = 2xy$



two solutions because $w = z^2$ maps two z's onto the same w. The map is 2 to 1.

The inverse map

$$z = w^{1/2}$$

is

$$re^{i\theta} = \sqrt{\rho}\, e^{i\phi/2} .$$

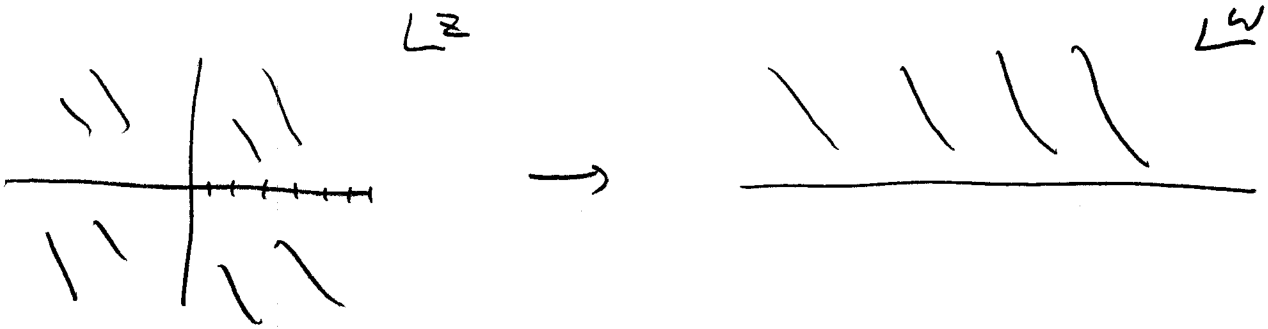
Now the whole w-plane $\rho \geq 0$ and $0 \leq \phi \leq 2\pi$ gets mapped onto the UHzP, since

$$0 \leq \theta = \phi/2 \leq \pi .$$

Equivalently

$$w = \rho e^{i\phi} = z^{1/2} = \sqrt{r}\, e^{i\theta/2} \qquad 0 \le \theta < 2\pi$$

maps the whole z-plane onto the UHwP.



But two w points, $\pm w$, correspond to the same z if we just mean that $w = \sqrt{z}$.

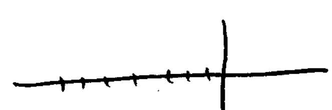
The definition $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$ puts a cut from 0 to ∞ on the $+$ real x-axis:

$$0 \le \theta < 2\pi .$$

Another possibility is to define $z^{1/2}$ as

$$z^{1/2} = \sqrt{r} \begin{cases} \theta/2 & \text{if } 0 \le \theta < \pi \\ -\theta/2 & \text{if } -\pi < \theta \le 0 \end{cases}$$

This definition puts the cut on the negative real axis



One also can say

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2} \quad \text{for} \quad 0 \le \theta < 4\pi$$

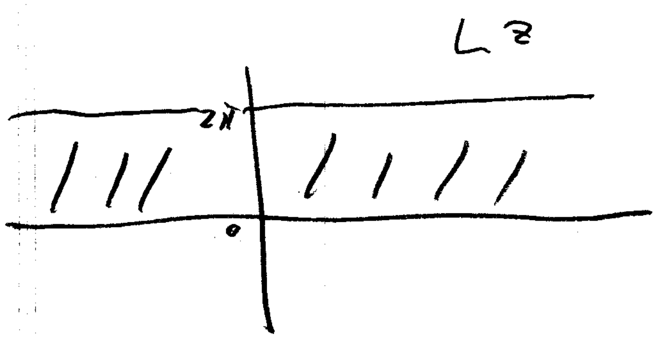
and now one covers the whole w-plane by covering the z-plane twice. This is a Riemann surface.

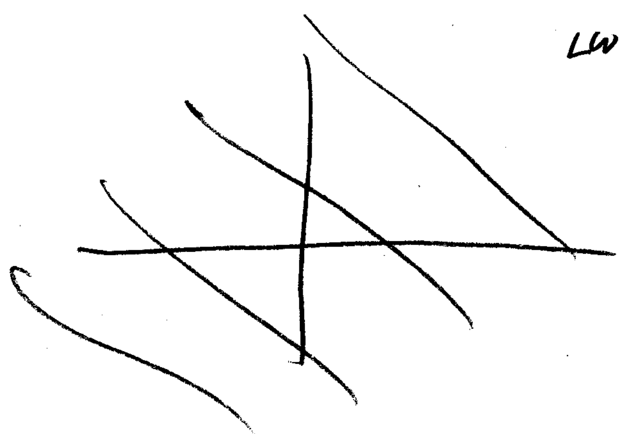
$$w = e^{z} \qquad \rho e^{i\phi} = e^{re^{i\theta}} = e^{x+iy}$$

so $\rho = e^{x}$ and $\phi = y$,

$$u + iv = e^{x+iy} = e^{x}(\cos y + i \sin y).$$

Now each band of the z-plane covers the whole w-plane





The inverse function is

$$w = \ln z = u + iv = \ln r e^{i\theta}$$