Final Examination in Physics 466

18 December 2003

(1) What is your name? Please **print** your name on your answer sheets. (10 points)

(2) Have you filled out your ICES form? (10 points)

All remaining problems are worth 20 points each. Do one problem from the set (3 – 8). All remaining problems are worth 20 points each.

(3) If S is the surface of a sphere of radius 2, what is the value of the surface integral

$$\frac{1}{3}\int_S \vec{r}\cdot d\vec{\sigma}\ ?$$

(4) If $\vec{A} = \vec{r}\times\vec{h}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$, what is the value of the surface integral

$$\int_S \vec{B}\cdot d\vec{\sigma}\ ?$$

S is the surface of problem (3). What if $\vec{A}$ is an arbitrary function — then what is the value of this surface integral?

(5) If u and v are scalar functions and

$$\vec{B} = (\vec{\nabla}u)\times(\vec{\nabla}v),$$

what is $\vec{\nabla}\cdot\vec{B}$?

(6) Let $\vec{V}(x,y,z)$ be the vector field $\vec{V}(x,y,z) = (3y^4z^2, 12xy^3z^2, 6xy^4z)$. The line integral

$$\int \vec{V}(x,y,z)\cdot(dx,dy,dz)$$

of $\vec{V}$ from the origin along the negative x axis to the point $(-1,0,0)$ and then in the y direction to the point $(-1,1,0)$ and then in the z direction to the point $(-1,1,1)$ is: (a) -1, (b) -2, (c) -3, (d) -4, (e) none of these.

(7) The surface integral $\int \nabla \times \vec{V} \cdot d\sigma$ of the curl of the same field $\vec{V}$ over the unit square in the x-y plane, i.e., over the set of points $(x, y, 0)$ with $0 \le x \le 1$ and $0 \le y \le 1$ is: (a) -1, (b) 0, (c) 1, (d) 2, (e) none of these. (Take $d\sigma$ to be parallel to $\hat{z}$.)

(8) Let $\vec{F}(x, y, z)$ be the vector field $\vec{F}(x, y, z) = (x^3, y^3, z^4)$. The surface integral $\int \vec{F} \cdot d\sigma$ of $\vec{F}$ over the unit cylinder $0 \le \rho \le 1$ and $0 \le z \le 1$ is: (a) 3.142, (b) 4.712, (c) 6.283, (d) 7.854, (e) none of these.

Now do one problem from the set (9 – 13).

(9) What are the eigenvalues of the matrix

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}?$$

(a)$(1 \pm \epsilon)^{\frac{1}{2}}$, (b) $1 \pm \epsilon$, (c) $(1 \pm \epsilon)^{-1}$, (d) $(1 \pm \epsilon)^{-\frac{1}{2}}$, (e) none of the above.

(10) Find the squared eigenvalues λ^2 and the eigenvalues λ of the helicity matrix

$$h = \frac{\vec{\sigma} \cdot \vec{p}}{|p|}$$

where the momentum $\vec{p}$ is a real three-vector and $|p| = \sqrt{\vec{p} \cdot \vec{p}}$ is its length. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(11) Suppose that $\{|n, 1\rangle\}$ and $\{|n, 2\rangle\}$ are two sets of complete, orthonormal vectors,

$$\langle n, 1|n', 1\rangle = \delta_{n,n'} \qquad \text{and} \qquad \langle n, 2|n', 2\rangle = \delta_{n,n'}.$$

Let $B = \sum_n |n, 1\rangle\langle n, 1|$, $C = \sum_n |n, 1\rangle\langle n, 2|$, $D = \sum_n |n, 2\rangle\langle n, 1|$, and $E = \sum_n |n, 2\rangle\langle n, 2|$. Let G be an arbitrary linear operator in the space spanned by the vectors $\{|n, 1\rangle\}$ and $\{|n, 2\rangle\}$. What are B^2, BG, $C^\dagger C$, $DD^\dagger$, GE, $B - E$, $B^2 - B$, and $E^2 - E$?

(12) One nice way of dealing with ordinary differential equations is to write them as first-order matrix equations. Thus for the case of the damped harmonic oscillator, one may express the two equations

$$\dot{x}(t) = v(t)$$

and

$$m\dot{v}(t) = -kx(t) - fv(t)$$

as the matrix equation

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -f/m \end{pmatrix}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

which we may integrate to

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = e^{At}\begin{pmatrix} x(0) \\ v(0) \end{pmatrix}$$

where A is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -k/m & -f/m \end{pmatrix}.$$

Find the eigenvalues and the normalized eigenvectors $|\pm\rangle$ of the matrix A. Compute the vectors $e^{At}|\pm\rangle$.

This method works very well for differential equations with constant coefficients. When the coefficients are functions of time, then one must time order the exponential

$$\mathcal{T}e^{\int_0^t A(t')dt'}.$$

(13) Consider three equal point masses, m, joined by two springs of equal spring constant, k. If the coordinates of the three masses are x_1, x_2, and x_3, then the lagrangian may be taken to be

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{k}{2}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2\right].$$

The equations of motion are Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i},$$

for $i = 1, 2, 3$. Find the normal modes of vibration and their natural frequencies. (The normal modes are the harmonic motions of the system.) Describe these modes both in words and by exhibiting their eigenvectors.

Now do one of the set (14 – 15):

(14) Indicate whether each of the following functions is an analytic function of the variable $z = x + iy$:

$$f(x, y) = x^2 + y^2$$

$$g(x,y) = x^2 - y^2 + 2ixy$$
$$h(x,y) = x^2 - y^2 - 2ixy.$$

(15) Compute the integral

$$I = \oint_C \frac{e^{imz}dz}{z^3 + z}$$

in which the contour C is counterclockwise about a circle $|z| = R$ of radius $R > 1$.

Now do one more problem — either (16) or one of the ones you skipped in the set (3 – 15).

(16) Consider a particle of mass m inside an impenetrable spherical shell of radius a. The wave function $\psi(r, \theta, \phi)$ will satisfy the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

where E is the energy. The boundary condition is that the wave function ψ vanish at $r = a$, which is the inner surface of the shell. By separating the variables, $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, you may see that the function $R(r)$ satisfies the radial equation

$$(r^2 R')' + [k^2 r^2 - l(l+1)]R = 0$$

where $l = 0, 1, 2, \ldots$ is an integer (the angular-momentum quantum number) and $k^2 = 2mE/\hbar^2$. The general solution of this equation is a linear combination of spherical Bessel functions of order l

$$R(r) = Aj_l(kr) + Bn_l(kr).$$

For the case of spherical symmetry $l = 0$ these functions are

$$j_0(kr) = \frac{\sin(kr)}{kr}$$

and

$$n_0(kr) = -\frac{\cos(kr)}{kr}.$$

Find for $l = 0$ the three lowest energy eigenvalues E.