Fourth HW Set

2.5.13 From Exercise 2.5.12 show that

$$-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\frac{\partial}{\partial\varphi}.$$

By Ex. (2.5.12)

$$x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x} = r\sin\theta\cos\phi\left[\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$-r\sin\theta\sin\phi\left[\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$=(\cos^2\phi+\sin^2\phi)\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\phi}$$

$$-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\frac{\partial}{\partial\phi}.$$

2.5.22 A magnetic vector potential is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}.$$

Show that this leads to the magnetic induction $\mathbf{B}$ of a point magnetic dipole with dipole moment $\mathbf{m}$.

ANS. for $\mathbf{m} = \hat{\mathbf{z}}m$,

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\frac{\mu_0}{4\pi}\frac{2m\cos\theta}{r^3} + \hat{\boldsymbol{\theta}}\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}.$$

Compare Eqs. (12.133) and (12.134)

$$B = \nabla \times A = \nabla \times \frac{\mu_0 m}{4\pi}\frac{\hat{z} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{m}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \partial_r & \partial_\theta & \partial_\varphi \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$

$$= \frac{\mu_0 m}{4\pi r^2 \sin\theta}\left[\hat{r}\,\partial_\theta \frac{\sin^2\theta}{r} - r\hat{\theta}\,\partial_r \frac{\sin^2\theta}{r}\right]$$

$$= \frac{\mu_0 m}{4\pi r^2 \sin\theta}\left[\frac{2\sin\theta\cos\theta}{r}\hat{r} + \frac{\sin\theta^2}{r}\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{4\pi}\left(\frac{2\cos\theta\,\hat{r}}{r^3} + \frac{\sin\theta\,\hat{\theta}}{r^3}\right).$$

2.5.23 At large distances from its source, electric dipole radiation has fields

$$\mathbf{E} = a_E \sin\theta \frac{e^{i(kr-\omega t)}}{r}\hat{\boldsymbol{\theta}}, \qquad \mathbf{B} = a_B \sin\theta \frac{e^{i(kr-\omega t)}}{r}\hat{\boldsymbol{\varphi}}.$$

Show that Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{B} = \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

are satisfied, if we take

$$\frac{a_E}{a_B} = \frac{\omega}{k} = c = (\varepsilon_0\mu_0)^{-1/2}.$$

Hint. Since r is large, terms of order r^{-2} may be dropped.

$$\vec{E} = a_E \sin\theta\, e^{i(kr-wt)} \frac{\hat{\theta}}{r} \qquad \vec{B} = a_B \sin\theta\, e^{i(kr-wt)} \frac{\hat{\phi}}{r}$$

$$\nabla \times E = \frac{a_E}{r^2\sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & \sin\theta e^{i(kr-wt)} & 0 \end{vmatrix}$$

$$= \frac{a_E}{r^2\sin\theta}\left[-\hat{r}\frac{\partial}{\partial\phi}\sin\theta e^{i(kr-wt)} + r\sin\theta\,\hat{\phi}\frac{\partial}{\partial r}\sin\theta e^{i(kr-wt)}\right]$$

$$= \frac{a_E}{r}\sin\theta\,\hat{\phi}\, ik\, e^{i(kr-wt)}$$

$$-\frac{\partial B}{\partial t} = i a_B w \sin\theta \frac{\hat{\phi}}{r} e^{i(kr-wt)}$$

2.3.23(2) So if $ka_E = a_B\omega$, ~~then~~

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

Now

$$\nabla\times B = \frac{a_B}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi}\\ \partial_r & \partial_\theta & \partial_\varphi\\ 0 & 0 & r\sin^2\theta\, e^{i(kr-\omega t)}\end{vmatrix}$$

$$= \frac{a_B}{r^2\sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}\sin^2\theta\, e^{i(kr-\omega t)} - r\hat{\theta}\frac{\partial}{\partial r}\sin^2\theta\, e^{i(kr-\omega t)}\right]$$

$$= \frac{a_B}{r^2\sin\theta}\left[\hat{r}\,2\sin\theta\cos\theta\, e^{i(kr-\omega t)} - i r\hat{\theta}\sin^2\theta\, k\, e^{i(kr-\omega t)}\right]$$

$$\approx -ik\frac{a_B\sin\theta}{r}e^{i(kr-\omega t)}\hat{\theta} \quad \text{neglecting } 1/r^2$$

$$\frac{1}{c^2}\frac{\partial E}{\partial t} = -i\frac{\omega}{c^2}a_E\frac{\sin\theta}{r}e^{i(kr-\omega t)}\hat{\theta} = \nabla\times B$$

as long as $ka_B = \omega c^{-2}a_E = \omega a_E\dfrac{a_B^2}{a_E^2} = \omega\dfrac{a_B^2}{a_E}$, i.e.,

$ka_E = \omega a_B$.

2.9.13 Show that the vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

(Exercise 1.5.12) follows directly from the description of a cross product with ε_{ijk} and the identity of Exercise 2.9.4.

$$\vec{A} \times \vec{B} \cdot \vec{C} \times \vec{D} = \epsilon_{ijk} A_j B_k \, \epsilon_{ilm} C_l D_m$$

$$= \left(\delta_{jl} \, \delta_{km} - \delta_{jm} \, \delta_{kl} \right) A_j B_k C_l D_m$$

$$= A \cdot C \;\; B \cdot D \; - \; A \cdot D \;\; B \cdot C$$

as long as the components all commute.

2.10.8 Evaluate $\partial \boldsymbol{\varepsilon}_i/\partial q^j$ for spherical polar coordinates, and from these results calculate Γ^k_{ij} for spherical polar coordinates.
Note. Exercise 2.5.1 offers a way of calculating the needed partial derivatives. Remember

$$\boldsymbol{\varepsilon}_1 = \hat{\mathbf{r}} \quad \text{but} \quad \boldsymbol{\varepsilon}_2 = r\hat{\boldsymbol{\theta}} \quad \text{and} \quad \boldsymbol{\varepsilon}_3 = r\sin\theta\hat{\boldsymbol{\varphi}}.$$

By Ex. (2.5.1)

$$\frac{\partial \epsilon_1}{\partial r} = \frac{\partial \hat{r}}{\partial r} = 0$$

$$\frac{\partial \epsilon_1}{\partial \theta} = \frac{\partial \hat{r}}{\partial \theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta = \hat{\theta}$$

$$\frac{\partial \epsilon_1}{\partial \phi} = \frac{\partial \hat{r}}{\partial \phi} = -\hat{x}\sin\theta\sin\phi + \hat{y}\sin\theta\cos\phi = \sin\theta\,\hat{\phi}$$

$$\frac{\partial \epsilon_2}{\partial r} = \frac{\partial r\hat{\theta}}{\partial r} = \hat{\theta} = \frac{\partial \epsilon_1}{\partial \theta}$$

$$\frac{\partial \epsilon_2}{\partial \theta} = \frac{\partial r\hat{\theta}}{\partial \theta} = r\left(-\hat{x}\sin\theta\cos\phi - \hat{y}\sin\theta\sin\phi - \hat{z}\cos\theta\right) = -\vec{r}$$

$$\frac{\partial \epsilon_2}{\partial \phi} = \frac{\partial r\hat{\theta}}{\partial \phi} = r\left(-\hat{x}\cos\theta\sin\phi + \hat{y}\cos\theta\cos\phi\right) = r\cos\theta\,\hat{\phi}$$

$$\frac{\partial \epsilon_3}{\partial r} = \frac{\partial(r\sin\theta\hat{\phi})}{\partial r} = \sin\theta\,\hat{\phi} = \frac{\partial \epsilon_1}{\partial \phi}$$

2.10.8 (2)

$$\frac{\partial \epsilon_3}{\partial \theta} = \frac{\partial (r \sin\theta \hat{\varphi})}{\partial \theta} = r \cos\theta \, \hat{\varphi} = \frac{\partial \epsilon_2}{\partial \varphi}$$

$$\frac{\partial \epsilon_3}{\partial \varphi} = \frac{\partial (r \sin\theta \hat{\varphi})}{\partial \varphi} = r \sin\theta \left(-\hat{x} \cos\varphi - \hat{y} \sin\varphi \right)$$

$$= -\vec{r} + \vec{z} = -r\hat{r} + \hat{r}\, r\cos\theta - r\hat{\theta} \sin\theta$$

By Eq. (2.131a), $\frac{\partial \epsilon_i}{\partial q^j} = \Gamma_{ij}^{k} \epsilon_k$, so

$\frac{\partial \epsilon_1}{\partial r} = 0 = \Gamma_{rr}^{k} \epsilon_k$, So $\Gamma_{rr}^{k} = 0 \quad k = r, \theta, \varphi.$

$\frac{\partial \epsilon_1}{\partial \theta} = \hat{\theta} = \frac{\epsilon_2}{r} = \Gamma_{r\theta}^{k} \epsilon_k$, So $\Gamma_{r\theta}^{\theta} = \frac{1}{r} = \Gamma_{\theta r}^{\theta}$

and $\Gamma_{r\theta}^{r} = \Gamma_{r\theta}^{\varphi} = 0 = \Gamma_{\theta r}^{r} = \Gamma_{\theta r}^{\varphi}$

$\frac{\partial \epsilon_1}{\partial \varphi} = \sin\theta \hat{\varphi} = \frac{\epsilon_\varphi}{r} = \Gamma_{r\varphi}^{k} \epsilon_k$, so $\Gamma_{r\varphi}^{\varphi} = \Gamma_{\varphi r}^{\varphi} = \frac{1}{r}$

and $\Gamma_{r\varphi}^{r} = \Gamma_{\varphi r}^{r} = \Gamma_{r\varphi}^{\theta} = \Gamma_{\varphi r}^{\theta} = 0.$

$\frac{\partial \epsilon_2}{\partial \theta} = -r\hat{r} = -r\epsilon_r = \Gamma_{\theta\theta}^{k} \epsilon_k$, so $\Gamma_{\theta\theta}^{r} = -r$

and $\Gamma_{\theta\theta}^{\theta} = \Gamma_{\theta\theta}^{\varphi} = 0.$

2.10.8(3)

$$\frac{\partial \epsilon_2}{\partial \phi} = r\cos\theta\hat{\phi} = \cot\theta\,\epsilon_\phi = \Gamma_{\theta\phi}^{k}\,\epsilon_k, \text{ so}$$

$$\Gamma_{\theta\phi}^{\phi} = \cot\theta = \Gamma_{\phi\theta}^{\phi}, \quad \text{while}$$

$$\Gamma_{\theta\phi}^{r} = \Gamma_{\phi\theta}^{r} = \Gamma_{\theta\phi}^{\theta} = \Gamma_{\phi\theta}^{\theta} = 0.$$

$$\frac{\partial \epsilon_3}{\partial \phi} = \frac{\partial \epsilon_\phi}{\partial \phi} = \hat{r}\,r(\cos\theta - 1) - r\sin\theta\,\hat{\theta} = \Gamma_{\phi\phi}^{k}\,\epsilon_k$$

So $\Gamma_{\phi\phi}^{r} = -r(1-\cos\theta)$ and

$\Gamma_{\phi\phi}^{\theta} = -\sin\theta$ and $\Gamma_{\phi\phi}^{\phi} = 0.$

2.10.11 From the circular cylindrical metric tensor g_{ij} calculate the Γ^k_{ij} for circular cylindrical coordinates.
Note. There are only three nonvanishing Γ.

$$\epsilon_\rho = \hat{\rho} = \frac{\partial \vec{r}}{\partial \rho} \qquad \epsilon_\varphi = r\hat{\varphi} = \frac{\partial \vec{r}}{\partial \varphi} \qquad \epsilon_z = \hat{z} = \frac{\partial \vec{r}}{\partial z}$$

$$g_{ij} = \epsilon_i \cdot \epsilon_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{so}$$

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr}\left(-\frac{\partial g_{\varphi\varphi}}{\partial r}\right) = -\frac{1}{2}\frac{\partial r^2}{\partial r} = -r$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(\frac{\partial g_{\varphi\varphi}}{\partial r}\right) = \frac{1}{2r^2}\, 2r = \frac{1}{r} = \Gamma^\varphi_{\varphi r}$$

$$\Gamma^z_{ij} = 0, \quad \Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \Gamma^r_{rz} = \Gamma^r_{zr} = \Gamma^r_{zz} = 0$$

$$\Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{rz} = \Gamma^\varphi_{zr} = \Gamma^\varphi_{zz} = \Gamma^\varphi_{rr} = 0$$

2.10.15 Show that for the metric tensor $g_{ij;k} = 0$, $g^{ij}_{;k} = 0$.

From (2.144), we may infer that the covariant derivative of a rank-2 covariant tensor is

$$(V_i W_k)_{;j} = V_i W_{k;j} + V_{i;j} W_k$$

$$= V_i \left(\partial_j W_k - W_\ell \Gamma^\ell_{kj} \right) + \left(\partial_j V_i - V_\ell \Gamma^\ell_{ij} \right) W_k$$

$$= \partial_j (V_i W_k) - V_i W_\ell \Gamma^\ell_{kj} - V_\ell W_k \Gamma^\ell_{ij}$$

because both sides are rank-3 covariant tensors. Thus

$$g_{ik;j} = \partial_j (g_{ik}) - g_{i\ell} \Gamma^\ell_{kj} - g_{\ell k} \Gamma^\ell_{ij} .$$

And from (2.141), we infer that

$$(V^i W^k)_{;j} = \partial_j (V^i W^k) + V^i W^\ell \Gamma^k_{\ell j} + V^\ell W^k \Gamma^i_{\ell j}$$

so that

$$g^{ik}{}_{;j} = \partial_j g^{ik} + g^{i\ell} \Gamma^k_{\ell j} + g^{\ell k} \Gamma^i_{\ell j} .$$

Now by (2.138), $\Gamma^\ell_{kj} = \frac{1}{2} g^{n\ell} \left(\partial_j g_{kn} + \partial_k g_{jn} - \partial_n g_{kj} \right)$

and $\Gamma^\ell_{ij} = \frac{1}{2} g^{n\ell} \left(\partial_j g_{in} + \partial_i g_{jn} - \partial_n g_{ij} \right)$, so

$$g_{ik;j} = \partial_j g_{ik} - \frac{1}{2} \delta_i^n \left(\partial_j g_{kn} + \partial_k g_{jn} - \partial_n g_{kj} \right)$$

$$- \frac{1}{2} \delta_k^n \left(\partial_j g_{in} + \partial_i g_{jn} - \partial_n g_{ij} \right)$$

2.10.15(2) So

$$g_{ik;j} = \partial_j g_{ik} - \frac{1}{2}\partial_j g_{ki} - \frac{1}{2}\partial_k g_{ji} + \frac{1}{2}\partial_i g_{kj}$$
$$- \frac{1}{2}\partial_j g_{ik} - \frac{1}{2}\partial_i g_{jk} + \frac{1}{2}\partial_k g_{ij} = 0$$

since $g_{ij} = g_{ji}$.

Although a direct approach is possible, it is easier to infer that

$$0 = (\delta^i_j)_{;n} = (g^{ik} g_{kj})_{;n}$$
$$= g^{ik}{}_{;n}\, g_{kj} + g^{ik}(g_{kj;n}) \quad \text{or}$$
$$0 = g_{kj}(g^{ik}{}_{;m}) \quad \text{or}$$
$$0 = g^{lk} g_{kj}\,(g^{ik}{}_{;m})$$
$$0 = \delta^l_j\, g^{ik}{}_{;n}. \quad \text{Set } l = j, \text{ then}$$
$$0 = g^{ik}{}_{;m}.$$

2.11.2 Starting with the divergence in tensor notation, Eq. (2.162), develop the divergence of a vector in spherical polar coordinates, Eq. (2.45).

By (2.162) $\quad \vec{\nabla}\cdot\vec{V} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^k}\left(\sqrt{g}\, V^k\right).$

Now

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad \text{so} \quad g = r^4\sin^2\theta$$

and $\sqrt{g} = r^2\sin\theta$. Now

$$V = V^k\epsilon_k = V_k\epsilon^k = V_r'\hat{r} + V_\theta'\hat{\theta} + V_\varphi'\hat{\varphi}$$

$$= V^r\hat{r} + V^\theta r\,\hat{\theta} + V^\varphi r\sin\theta\,\hat{\varphi}$$

where the primes mean the notation of Eq. (2.45). So

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta V_r'\right) + \frac{\partial}{\partial\theta}\left(r^2\sin\theta\frac{V_\theta'}{r}\right) + \frac{\partial}{\partial\varphi}\left(r^2\sin\theta\frac{V_\varphi'}{r\sin\theta}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2V_r'\right) + r\frac{\partial}{\partial\theta}\left(\sin\theta V_\theta'\right) + r\frac{\partial V_\varphi'}{\partial\varphi}\right].$$

2.11.3 The covariant vector A_i is the gradient of a scalar. Show that the difference of covariant derivatives $A_{i;j} - A_{j;i}$ vanishes.

By (2.164), the covariant curl is the curl:

$$A_{i;j} - A_{j;i} = \frac{\partial A_i}{\partial q^j} - \frac{\partial A_j}{\partial q^i},$$

so if $A_i = \frac{\partial s}{\partial q^i}$, then

$$A_{i;j} - A_{j;i} = \frac{\partial^2 s}{\partial q^j \partial q^i} - \frac{\partial^2 s}{\partial q^i \partial q^j} = 0.$$

Fifth Set of HW Solutions

3.2.2 Show that

$$(\mathsf{A}+\mathsf{B})(\mathsf{A}-\mathsf{B}) = \mathsf{A}^2 - \mathsf{B}^2$$

iff A and $\mathbf{B}$ **commute,**

$$[\mathsf{A}, \mathsf{B}] = 0.$$

3.2.2

$$(A+B)(A-B) = A^2 - B^2 + BA - AB$$

$$= A^2 - B^2$$

iff

$$BA - AB = 0 = [B, A].$$

3.2.7 Given the three matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

find all possible products of A, B, and C, two at a time, including squares. Express your answers in terms of A, B, and C, and 1, the unit matrix. These three matrices together with the unit matrix form a representation of a mathematical group, the **vierergruppe** (see Chapter 4; vier is 4 in German).

3.2.7 $A^2 = 1, \quad B^2 = 1, \quad C^2 = 1$

$$AB = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = C$$

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = C$$

$$BC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A$$

$$CB = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A$$

$$AC = -AB = -C = B$$

$$CA = -BA = -C = B.$$

3.2.9 Verify the **Jacobi identity**

$$[A, [B, C]] = [B, [A, C]] - [C, [A, B]].$$

This is useful in matrix descriptions of elementary particles (see Eq. (4.16)). As a **mnemonic aid, the reader might note that the Jacobi identity** has the same form as the $BAC - CAB$ rule of Section 1.5.

3.2.9

$$[A, [B, C]] + [B, [C, A] + [C, [A, B]] =$$

$$A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B$$

$$+ C(AB - BA) - (AB - BA)C$$

$$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA} + \cancel{BCA} - BAC$$

$$- \boxed{CAB} + \cancel{ACB} + \boxed{CAB} \cancel{- CBA} - \cancel{ABC} + BAC$$

$$= 0$$

3.2.13 The three Pauli spin matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that

(a) $\sigma_i^2 = 1$,
(b) $\sigma_i \sigma_j = i\sigma_k$, $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ (cyclic permutation),
(c) $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1_2$, where 1_2 is the 2×2 unit matrix.

These matrices were used by Pauli in the nonrelativistic theory of electron spin.

3.2.13 (a) $\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$

(b) $\sigma_1 \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$

$\sigma_2 \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1$

$\sigma_3 \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2$

(c) $\sigma_2 \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3$

$\sigma_3 \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1$

$\sigma_1 \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_2$

So

$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} I$, Better yet

$$\sigma_i \sigma_j = \delta_{ij} + i\sum_{k=1}^{3} \epsilon_{ijk} \sigma_k .$$

3.2.14 Using the Pauli σ of Exercise 3.2.13, show that

$$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}\ 1_2 + i\sigma \cdot (\mathbf{a} \times \mathbf{b}).$$

Here

$$\sigma \equiv \hat{\mathbf{x}}\sigma_1 + \hat{\mathbf{y}}\sigma_2 + \hat{\mathbf{z}}\sigma_3,$$

a and **b** are ordinary vectors, and 1_2 is the 2×2 unit matrix.

3.2.14 $\sigma \cdot a\ \sigma \cdot b = \sum a_i b_j\ \sigma_i \sigma_j$

$= \sum a_i b_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$

$= a \cdot b + i(a \times b) \cdot \sigma$

3.2.15 One description of spin 1 particles uses the matrices

$$\mathsf{M}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{M}_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

and

$$\mathsf{M}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Show that

(a) $[\mathsf{M}_x, \mathsf{M}_y] = i\mathsf{M}_z$, and so on[12] (cyclic permutation of indices). Using the Levi-Civita symbol of Section 3.4, we may write

$$[\mathsf{M}_i, \mathsf{M}_j] = i\varepsilon_{ijk}\mathsf{M}_k.$$

(b) $\mathsf{M}^2 \equiv \mathsf{M}_x^2 + \mathsf{M}_y^2 + \mathsf{M}_z^2 = 2\ 1_3$,
where 1_3 is the 3×3 unit matrix.

(c) $[\mathsf{M}^2, \mathsf{M}_i] = 0$,
$[\mathsf{M}_z, \mathsf{L}^+] = \mathsf{L}^+$,
$[\mathsf{L}^+, \mathsf{L}^-] = 2\mathsf{M}_z$,
where
$\mathsf{L}^+ \equiv \mathsf{M}_x + i\mathsf{M}_y$,
$\mathsf{L}^- \equiv \mathsf{M}_x - i\mathsf{M}_y$.

3.2.15 (a)

$$M_1 M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}$$

$$M_2 M_1 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}$$

$$[M_1, M_2] = \frac{1}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = i\,\epsilon_{123} M_3.$$

$$M_2 M_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_3 M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

3.2.15 (a)

$$[M_2, M_3] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} = i\epsilon_{231} M_1$$

$$M_3 M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_1 M_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[M_3, M_1] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= i\epsilon_{312} M_2 .$$

(b) $$M_1^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_2^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$M_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_1^2 + M_2^2 + M_3^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I = \vec{M}^2.$$

c) $$[M^2, M_i] = [2I, M_i] = 0$$

3.2.15(3)

$$[M_3, L^+] = [M_3, M_1 + iM_2] = iM_2 + M_1 = L^+$$

$$[L^+, L^-] = [M_1 + iM_2, M_1 - iM_2] = -2i[M_1, M_2] = 2M_3$$

3.2.16 Repeat Exercise 3.2.15 using an alternate representation,

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad M_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

and

$$M_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In Chapter 4 these matrices appear as the **generators** of the rotation group.

(a) $M^b_{ac} = i\,\epsilon_{abc}$. So $\left([M^b, M^d]\right)_{ac} = i\epsilon_{abe}\, i\epsilon_{cde} + \epsilon_{adc}\epsilon_{cbe}$

$= \delta_{ab}\delta_{de} - \delta_{ae}\delta_{db} - (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) = \delta_{ab}\delta_{de} - \delta_{ad}\delta_{be}$

$= \delta_{ba}\delta_{de} - \delta_{be}\delta_{da} = \epsilon_{bdc}\epsilon_{cae} = i\,\epsilon_{bdc}\, i\,\epsilon_{ace}$

$= i\,\epsilon_{bdc} M^c_{ae}$. So $[M^b, M^d] = i\,\epsilon_{bdc} M^c$.

(b)

$$M_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_1^2 + M_2^2 + M_3^2 = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2I$$

(c) $[M^2, M_i] = [2I, M_i] = 0$

$[M_3, L^+] =$

$[M_3, M_1 + iM_2] = i\epsilon_{312}M_2 + i^2\epsilon_{321}M_1 = M_1 + iM_2 = L^+$

$[L^+, L^-] = [M_1 + iM_2, M_1 - iM_2] = i^2\epsilon_{213}M_3 - i\,i\,\epsilon_{123}M_3 =$

$= M_3 + M_3 = 2M_3$

8.3.4 Verify that

$$\nabla^2\psi(r,\theta,\varphi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi)\right]\psi(r,\theta,\varphi) = 0$$

is separable (in spherical polar coordinates). The functions $f, g,$ and h are functions only of the variables indicated; k^2 is a constant.

8.3.4 $\quad \psi = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

$$\nabla^2 R(r)\Theta(\theta)\Phi(\varphi) + \left[k^2 + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2\sin^2\theta}\right]R\Theta\Phi = 0$$

$$= \frac{1}{r^2\sin\theta}\left[\Theta\Phi\sin\theta\,(r^2R')' + R\Phi(\sin\theta\,\Theta')' + \frac{R\Theta}{\sin\theta}\Phi''\right]$$

$$+ \left[k^2 + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2\sin^2\theta}\right]R\Theta\Phi.$$ Divide by $R\Theta\Phi$

$$0 = \frac{(r^2R')'}{r^2R} + \frac{(\sin\theta\,\Theta')'}{r^2\sin\theta\,\Theta} + \frac{\Phi''}{r^2\sin^2\theta\,\Phi}$$

$$+ k^2 + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2\sin^2\theta}$$

$$\frac{(r^2R')'}{R} + r^2k^2 + r^2f(r) = -\frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta} - \frac{\Phi''}{\sin^2\theta\,\Phi}$$

$$- g(\theta) - \frac{h(\varphi)}{\sin^2\theta} = A$$

So

$$(r^2R')' + r^2k^2 + r^2f(r) = AR.$$

8.3.4(2)

$$+\frac{\sin\theta\,(\sin\theta\,\Theta')'}{\Theta} + g(\theta)\sin^2\theta + A\sin^2\theta$$

$$= -\frac{\Phi''}{\Phi} - h(\varphi) = B$$

S.

$$\sin\theta\,(\sin\theta\,\Theta')' + g(\theta)\sin^2\theta\,\Theta + A\sin^2\theta\,\Theta = B\Theta$$

and

$$-\phi'' = B\phi + \phi h(\varphi).$$

8.3.6 For a homogeneous spherical solid with constant thermal diffusivity K and no heat sources the equation of heat conduction becomes

$$\frac{\partial T(r,t)}{\partial t} = K\nabla^2 T(r,t).$$

Assume a solution of the form

$$T = R(r)T(t)$$

and separate variables. Show that the radial equation may take on the standard form

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [\alpha^2 r^2 - n(n+1)]R = 0; \quad n = \text{integer}.$$

The solutions of this equation are called spherical Bessel functions.

8.3.6 We set $T(\vec{x},t) = R(r)Y_{em}(\theta,\varphi)\,T(t)$

and use Ex. 2.5.16

$$-\Delta = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{r^2} \quad \text{in which}$$

$$L^2 Y_{em}(\theta,\varphi) = \ell(\ell+1)Y_{\ell m}, \quad \text{So}$$

$$\frac{\partial T(x,t)}{\partial t} = RYT' = K\Delta RYT = KT\Delta RY$$

$$= KT\left[Y_{em}\frac{1}{r^2}(r^2R')' - \frac{R}{r^2}\ell(\ell+1)Y_{em}\right]$$

So dividing by $RTY_{\ell m}$, we get

$$\frac{T'}{T} = K\left[\frac{1}{Rr^2}(r^2R')' - \frac{\ell(\ell+1)}{r^2}\right] = -C$$

$$r^2R'' + 2rR' + [\alpha^2r^2 - \ell(\ell+1)]R = 0 \quad \text{with} \quad \alpha^2 = C/K.$$

8.3.8 The quantum mechanical angular momentum operator is given by $L = -i(\mathbf{r} \times \nabla)$. Show that

$$\mathbf{L} \cdot \mathbf{L}\psi = l(l+1)\psi$$

leads to the associated Legendre equation.
Hint. Exercises 1.9.9 and 2.5.16 may be helpful.

8.3.8 By Ex. (2.5.16.c),

$$L^2\psi = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right) - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = l(l+1)\psi$$

We set

$$\psi = \Theta(\theta)\, e^{im\varphi} \quad \text{and set}$$

$$-\frac{e^{im\varphi}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{m^2 e^{im\varphi}}{\sin^2\theta}\Theta = l(l+1)e^{im\varphi}\Theta$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta$$

which is Eq. (12.80).

Extra-Credit Problem: Show that if

$$[X_a, X_b] = i f_{abd} X_d \qquad (1)$$

with f_{abd} real and totally anti-symmetric, then the matrices

$$(T_b)_{ac} = i f_{abc} \qquad (2)$$

satisfy the commutation relation

$$[T_a, T_b] = i f_{abc} T_c. \qquad (3)$$

(The summation over repeated indices is implied.)
By (1), we have

$$[X_a, [X_b, X_c]] = [X_a, i f_{bcd} X_d]$$

$$= - f_{bcd} f_{ade} X_e. \qquad (4)$$

But now the Jacobi identity

$$0 = [X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] \qquad (5)$$

implies

$$0 = f_{bcd} f_{ade} X_e + f_{cad} f_{bde} X_e + f_{abd} f_{cde} X_e. \qquad (6)$$

Since the matrices X_e are linearly independent, we have

$$0 = f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} \qquad (7)$$

or via (2)

$$0 = +(T_a)_{cd} (T_b)_{de} - (T_b)_{cd} (T_a)_{de} - i f_{abd} (T_d)_{ce} \qquad (8)$$

or

$$[T_a, T_b] = i f_{abd} T_d \qquad (9)$$

which is (3).