1.7.6 $$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = k\frac{\vec{r}}{r^3}$$

$$\nabla \cdot E = \sum_{i=1}^{3} \frac{\partial E_i}{\partial x_i} = k\left(\frac{3}{r^3} - \frac{3}{2}\frac{\vec{r}\cdot 2\vec{r}}{r^5}\right) = 0$$

So if $r \neq 0$, $\nabla \cdot E = 0$.

At $r = 0$, $\nabla \cdot E$ is singular and not defined as a function. As a distribution, it is

$$\nabla \cdot E(\vec{r}) = \frac{q}{\epsilon_0}\,\delta^3(\vec{r}).$$

1.8.7

$$L_i = (r \times p)_i = \sum_{jk=1}^{3} \epsilon_{ijk} \, r_j \, p_k$$

$$= \frac{\hbar}{i} \sum_{j,k} \epsilon_{ijk} \, r_j \nabla_k$$

1.8.8 summation convention in use:

$$[L_i, L_j] = -\hbar^2 [\epsilon_{ik\ell} r_k \nabla_\ell, \epsilon_{jnm} r_n \nabla_m]$$

$$= -\hbar^2 \big(\delta_{\ell n} \epsilon_{ik\ell} \epsilon_{jnm} r_k \nabla_m$$
$$- \delta_{mk} \epsilon_{ik\ell} \epsilon_{jnm} r_n \nabla_\ell \big)$$

$$= -\hbar^2 \big(\epsilon_{ik\ell} \epsilon_{j\ell m} r_k \nabla_m - \epsilon_{ik\ell} \epsilon_{jnk} r_n \nabla_\ell \big)$$

$$= \hbar^2 \big(\delta_{ij} \delta_{km} - \delta_{im} \delta_{kj} \big) r_k \nabla_m$$

$$- \hbar^2 \big(\delta_{ij} \delta_{\ell n} - \delta_{in} \delta_{\ell j} \big) r_n \nabla_\ell$$

$$= \hbar^2 \big(\delta_{ij} \mathbf{r} \cdot \nabla - r_j \nabla_i - \delta_{ij} \mathbf{r} \cdot \nabla + r_i \nabla_j \big)$$

$$= \hbar^2 (r_i \nabla_j - r_j \nabla_i)$$

But $i\hbar \epsilon_{ijk} L_k = i\hbar \epsilon_{ijk} \frac{\hbar}{i} \epsilon_{k\ell m} r_\ell \nabla_m$

$$= \hbar^2 (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) r_\ell \nabla_m$$

$$= \hbar^2 (r_i \nabla_j - r_j \nabla_i).$$ Thus

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k.$$

Now

$$(L \times L)_i = \epsilon_{ijk} L_j L_k$$

$$= \epsilon_{ijk} \left(\tfrac{1}{2}(L_j L_k + L_k L_j) + \tfrac{1}{2}[L_j, L_k] \right)$$

$$= \tfrac{1}{2}\epsilon_{ijk}\, i\hbar\, \epsilon_{jkl} L_l$$

$$= i\hbar L_i \qquad \text{or}$$

$$\vec{L} \times \vec{L} = i\hbar \vec{L}.$$

1.8.9

$$[a\cdot L, b\cdot L] = \sum_{ij} a_i b_j [L_i, L_j]$$

$$= \sum_{ij} i\hbar\, a_i b_j \epsilon_{ijk} L_k$$

$$= i\hbar\, (a\times b)\cdot L$$

1.8.15 $\nabla \cdot B = \nabla \times B = 0$ $\nabla_i m_j = 0$

$F = \nabla \times (B \times m)$

$F_i = \epsilon_{ijk} \nabla_j \epsilon_{klm} B_l m_m$

$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})(\nabla_j B_l) m_m$

$= (\nabla_j B_i) m_j - (\nabla \cdot B) m_i$

$= (\nabla_j B_i) m_j = (\nabla_i B_j) m_j$

$= \nabla_i (B_j m_j) = \nabla_i (B \cdot m)$

in which $\nabla_i B_j = \nabla_j B_i$ because $\nabla \times B = 0$. Thus

$$\vec{F} = \vec{\nabla} (\vec{B} \cdot \vec{m}).$$

1.8.16 $\phi = \dfrac{p \cdot r}{4\pi\epsilon_0 r^3}$

$$E_i = -\frac{\partial \phi}{\partial r_i} = -\frac{p_i}{4\pi\epsilon_0 r^3} + \frac{3}{2}\frac{p \cdot r}{4\pi\epsilon_0}\frac{2r_i}{r^5}$$

$$= \frac{1}{4\pi\epsilon_0}\left(3\frac{p \cdot r \, r_i}{r^5} - \frac{p_i}{r^3} \right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3 p \cdot r \, \vec{r} - \vec{p} r^2}{r^5} \right).$$

1.9.12

If $\nabla \times (\nabla \times A) = k^2 A$, then

$$0 = \nabla \cdot \big(\nabla \times (\nabla \times A)\big) = k^2 \; \nabla \cdot A.$$

So

$$\nabla \cdot A = 0 \qquad \text{if} \quad \vec{k} \neq \vec{0},$$

By 1.86,

$$\nabla \times (\nabla \times A) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = k^2 \vec{A}$$

But $\nabla \cdot A = 0$. So

$$-\Delta \vec{A} = k^2 \vec{A}, \qquad \text{or}$$

$$\Delta \vec{A} + k^2 \vec{A} = 0$$

1.10.4

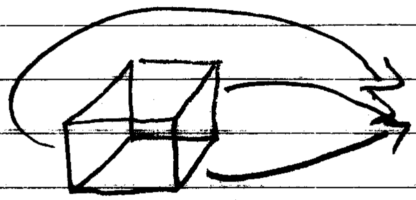
$$(\vec{\nabla} \times \vec{r})_i = \epsilon_{ijk} \partial_j r_k = 0$$

So $\oint \vec{r} \cdot d\vec{r} = \int (\nabla \times r) \cdot ds = 0.$

1.10.5

$$\frac{1}{3}\int \mathbf{r}\cdot d\mathbf{s} = \frac{1}{3}\int \nabla\cdot\mathbf{r}\, dv$$

$$= \int dv = 1.$$



$\int \mathbf{r}\cdot d\mathbf{s} = 0$ for each of these

$$\frac{1}{3}\int_{top} \mathbf{r}\cdot d\mathbf{s} = \frac{1}{3}\int \hat{z}\, ds = \frac{1}{3}$$

$$\frac{1}{3}\int_{front} \mathbf{r}\cdot d\mathbf{s} = \frac{1}{3}\int \hat{x}\, ds = \frac{1}{3}$$

$$\frac{1}{3}\int_{R} \mathbf{r}\cdot d\mathbf{s} = \frac{1}{3}\int \hat{y}\, ds = \frac{1}{3}$$

So explicit integration verifies this application of Gauss's theorem.

1.11.7

$$\epsilon_0 \int E^2 dv = \epsilon_0 \int (\nabla\phi)^2 dv$$

$$= -\epsilon_0 \int \phi \Delta \phi \, dv = \int \phi \rho \, dv$$

since $-\epsilon_0 \Delta\phi = \epsilon_0 \nabla \cdot E = \rho$.

1.11.9

$$W = \frac{1}{2} \int H \cdot B \, dv = \frac{1}{2} \int H \cdot \nabla \times A \, dv$$

$$= \frac{1}{2} \int H_i \epsilon_{ijk} \partial_j A_k \, dv$$

$$= -\frac{1}{2} \int A_k \epsilon_{ijk} \partial_j H_i \, dv$$

$$= \frac{1}{2} \int A \cdot \nabla \times H \, dv .$$

But $\nabla \times H = \frac{4\pi}{c} J + \frac{\dot{D}}{c} \rightarrow \frac{4\pi J}{c}$

$$W = \frac{2\pi}{c} \int A \cdot J \, dv.$$

In RMKS $\qquad \nabla \times H = J + \dot{D} \rightarrow J$

and then

$$W = \frac{1}{2} \int A \cdot J \, dv$$

1.12.5

$$\oint H \cdot dr = \int \nabla \times H \cdot ds$$

$$= \int \tilde{J} \cdot ds = I.$$

1.13.2

$$\int E \cdot dS = \int \nabla \cdot E \, dv = \int 4\pi\rho \, dv = 4\pi Q$$

$$\parallel$$

$$4\pi E r^2$$

So $\vec{E} = \frac{Q}{r^2} \hat{r} = -\vec{\nabla}\phi$

So

$$\phi(r) - \phi(r_0) = \int_{r_0}^{r} \vec{\nabla}\phi \cdot dr' = \int_{r_0}^{r} - E \cdot dr'$$

$$= -Q \int_{r_0}^{r} \frac{dr'}{r'^2} = Q \left[\frac{1}{r}\right]_{r_0}^{r} = Q\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

Send $r_0 \to \infty$, assume $\phi(r_0) = 0$. Then

$$\phi(\vec{r}) = \frac{Q}{r}.$$

1.13.5 By recipe (1.140),

$$\vec{A}(x,y,z) = \hat{z}\left[\int_{y_0}^{y} B_1(x_0,y',z)\,dy' - \int_{x_0}^{x} B_2(x',y,z)\,dx'\right]$$

$$= \hat{z}\left[\int_{y_0}^{y} -\frac{\mu_0 I}{2\pi}\frac{y'}{x_0^2+y'^2}\,dy - \frac{\mu_0 I}{2\pi}\int_{x_0}^{x}\frac{x'}{x'^2+y^2}\,dx'\right]$$

$$= -\frac{\mu_0 I}{2\pi}\hat{z}\left[\frac{1}{2}\left|\log(x_0^2+y'^2)\right|_{y_0}^{y} + \frac{1}{2}\left|\log(x'^2+y^2)\right|_{x_0}^{x}\right]$$

$$= -\frac{\mu_0 I}{4\pi}\hat{z}\left[\log\frac{x_0^2+y^2}{x_0^2+y_0^2} + \log\frac{x^2+y^2}{x_0^2+y^2}\right]$$

$$= -\frac{\mu_0 I}{4\pi}\hat{z}\,\log\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right)$$

Adding to $\vec{A}$ a term that is independent of x, y, z, and t is immaterial.

1.13.8 $\quad B_i = \epsilon_{ijk}\, \partial_j u\, \partial_k v$

(a) So $\quad \partial_i B_i = \epsilon_{ijk}\, \partial_i \partial_j u\, \partial_k v + \epsilon_{ijk}\, \partial_j u\, \partial_i \partial_k v$

$$= 0$$

because it is a sum of terms that are products of an anti-symmetric factor and a symmetric factor.

(b) $\quad (\nabla \times A)_i = \epsilon_{ijk}\, \partial_j A_k$

$$= \epsilon_{ijk}\, \partial_j \tfrac{1}{2}(u\, \partial_k v - v\, \partial_k u)$$

$$= \tfrac{1}{2}\, \epsilon_{ijk} (\partial_j u\, \partial_k v - \partial_j v\, \partial_k u)$$

$$= \tfrac{1}{2}\, \epsilon_{ijk} (\partial_j u\, \partial_k v - \partial_k u\, \partial_j v)$$

$$= \epsilon_{ijk}\, \partial_j u\, \partial_k v = (\nabla u \times \nabla v)_i$$

$$= B_i .$$

1.13.9

First $B' = \nabla \times A' = \nabla \times (A + \nabla \phi)$

$$= \nabla \times A = B.$$

so

$$\int \vec{B}' \cdot \vec{ds} = \int \vec{B} \cdot \vec{ds}.$$

$$\oint A' \cdot dr = \oint (A + \nabla \phi) \cdot dr$$

$$= \oint A \cdot dr + \oint \nabla \phi \cdot dr$$

$$= \oint A \cdot dr + \phi(r_0) - \phi(r_0)$$

$$= \oint A \cdot dr.$$

1.13.10

$$\nabla \times \left(E + \frac{\partial A}{c\partial t}\right) = \nabla \times E + \frac{\partial}{c\partial t}(\nabla \times A)$$

$$= \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{c\partial t} = 0$$

by a Maxwell equation.

So by (1.117) – (1.129), we may set

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$

in any simply connected region.

1.14.4 The Maxwell equation

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t}$$

becomes, if the system is stationary,

$$\nabla \times B = \frac{4\pi}{c} J.$$

If $B = \nabla \times A$, then

$$\nabla \times (\nabla \times A) = \frac{4\pi}{c} J.$$

By (1.86) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \cdot \vec{\nabla} \vec{A}$, so

$$\vec{\nabla}(\nabla \cdot A) - \nabla \cdot \nabla \vec{A} = \frac{4\pi}{c} \vec{J}.$$

If we now use the gauge $\vec{\nabla} \cdot \vec{A} = 0$, then

$$\nabla \cdot \nabla \vec{A} = -\frac{4\pi}{c} \vec{J}.$$

That is, $$\nabla^2 \vec{A} = \Delta \vec{A} = -\frac{4\pi}{c} \vec{J}.$$

1.15.4 We assume that $f(x)$ is continuously differentiable and that

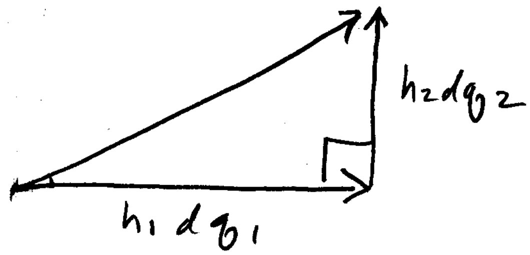
$$\lim_{x \to \pm\infty} f(x) = 0.$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \frac{\sin nx}{\pi x} dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f\left(\frac{y}{n}\right) \frac{\sin y}{\pi y} \frac{n\,dy}{n}$$

$$= \int_{-\infty}^{\infty} \lim_{n\to\infty} f\left(\frac{y}{n}\right) \frac{\sin y}{\pi y} dy$$

$$= f(0) \int_{-\infty}^{\infty} \frac{\sin y}{\pi y} dy = f(0) \text{ by } (7.51).$$

2.1.1



$$ds^2 = g_{11} dq_1^2 + g_{22} dq_2^2 + 2 g_{12} dq_1 dq_2$$
$$= h_1^2 dq_1^2 + h_2^2 dq_2^2 + 2 h_1 h_2 dq_1 dq_2$$
$$= h_1^2 dq_1^2 + h_2^2 dq_2^2 + 2 h_1 h_2 dq_1 dq_2 \cos \frac{\pi}{2}$$
$$= h_1^2 dq_1^2 + h_2^2 dq_2^2$$

So $g_{12} = 0$. Similarly, one may show that all the other off-diagonal terms g_{ij} vanish.

2.2.2 $\quad \vec{V} = \vec{e}_1$ means $V_1 = 1,\ V_2 = V_3 = 0,$
So

(a) $$\nabla \cdot e_1 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3)\right] = \frac{1}{h_1 h_2 h_3}\frac{\partial(h_2 h_3)}{\partial q_1}.$$

(b) $$\nabla \times e_1 = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} e_1 h_1 & e_2 h_2 & e_3 h_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{h_1 h_2 h_3}\left[-e_2 h_2(-\partial_3 h_1) + e_3 h_3(-\partial_2 h_1)\right]$$

$$= \frac{1}{h_1}\left[\frac{\vec{e}_2}{h_3}\frac{\partial h_1}{\partial q_3} - \frac{\vec{e}_3}{h_2}\frac{\partial h_1}{\partial q_2}\right].$$

2.4.12

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ 0 & 0 & -\frac{\mu I}{2\pi} \ln\rho \end{vmatrix}$$

$$= -\frac{\mu I}{2\pi\rho}\left[\hat{\rho}\frac{\partial}{\partial\phi}(\ln\rho) - \rho\hat{\phi}\frac{\partial}{\partial\rho}\ln\rho\right]$$

$$= \frac{\mu I}{2\pi\rho}\hat{\phi}\rho\frac{1}{\rho} = \frac{\mu I}{2\pi\rho}\hat{\phi}.$$

2.4.14 (b)

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho E_\rho\right) = E_0 a e^{i(kz-\omega t)} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho}{\rho}\right)$$

$$= E_0 a e^{i(kz-\omega t)} \frac{1}{\rho}\frac{\partial}{\partial\rho}(1) = 0,$$

which is right since $\rho = 0$ inside cable.

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} = B_0 a e^{i(kz-\omega t)} \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{1}{\rho} = 0,$$

$$\nabla\times E = \frac{aE_0}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ \frac{e^{i(kz-\omega t)}}{\rho} & 0 & 0 \end{vmatrix}$$

$$= \frac{aE_0}{\rho} e^{-i\omega t}\left[-\rho\hat{\phi}\left(-\frac{\partial}{\partial z}\right)\frac{e^{ikz}}{\rho} + \hat{z}\frac{\partial}{\partial\phi}\frac{e^{ikz}}{\rho}\right]$$

$$= aE_0 e^{-i\omega t}\frac{\hat{\phi}\, ik e^{ikz}}{\rho} = ik\, aE_0 \frac{e^{i(kz-\omega t)}}{\rho}\hat{\phi}.$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left[\hat{\phi}\, B_0\frac{a}{\rho}e^{i(kz-\omega t)}\right] = -i\frac{\omega}{c}\hat{\phi}\, B_0\frac{a}{\rho}e^{i(kz-\omega t)}$$

In my units, $k = \frac{\omega}{c}$ and $E_0 = B_0$.

2.4.14

$$\vec{E} = \hat{\rho} E_0 \left(\frac{a}{\rho}\right) e^{i(kz-wt)} \quad \& \quad \vec{B} = \hat{\phi} B_0 \frac{a}{\rho} e^{i(kz-wt)}$$

(a) By (2.35),

$$\nabla^2 \vec{E}\Big|_\rho = \nabla^2 E_\rho - \frac{1}{\rho^2} E_\rho = E_0 a\, e^{-iwt}\left(\nabla^2 \frac{e^{ikz}}{\rho} - \frac{e^{ikz}}{\rho^3}\right).$$

$$\nabla^2 \frac{e^{ikz}}{\rho} = \frac{e^{ikz}}{\rho}\partial_\rho\left(\rho\, \partial \frac{1}{\rho}\right) - \frac{k^2 e^{ikz}}{\rho} = \frac{e^{ikz}}{\rho}\frac{d}{d\rho}\left(\rho\left(-\frac{1}{\rho^2}\right)\right) - \frac{k^2 e^{ikz}}{\rho} \qquad \text{by (2.33)}$$

$$= \frac{e^{ikz}}{\rho^3} - \frac{k^2 e^{ikz}}{\rho}.$$ The ρ^{-3} terms cancel and

So $\nabla^2 \vec{E}\,|_\rho = -k^2 \hat{\rho} E_\rho$. $\nabla^2 \vec{E}\,|_\phi = \nabla^2 E\,|_z = 0$.

So

$$\nabla^2 \vec{E} = -k^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$ — the wave equation.

The book is wrong here: $\Delta \vec{E} \neq 0$; $\Delta \vec{B} \neq 0$.

$$\nabla^2 B\,|_\rho = -\frac{2}{\rho^2}\frac{\partial B_\phi}{\partial \phi} = 0$$

$$\nabla^2 B\,|_\phi = \nabla^2 B_\phi - \frac{1}{\rho^2} B_\phi = B_0 a\, e^{i(kz-wt)}\left(\nabla^2 \frac{1}{\rho} - \frac{1}{\rho^3}\right) - k^2 B_\phi$$

So $\nabla^2 \vec{B} = -k^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$, which is the vacuum wave equation.

2.4.14 (b) So

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0.$$

$$\nabla \times B = \frac{aB_0 e^{-i\omega t}}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ 0 & e^{ikz} & 0 \end{vmatrix}$$

$$= \frac{aB_0 e^{-i\omega t}}{\rho} \left[\hat{\rho}\left(-\frac{\partial}{\partial z}\right) e^{ikz} + \hat{z}\frac{\partial}{\partial \rho} e^{ikz} \right]$$

$$= \frac{aB_0 e^{-i\omega t}}{\rho} \left(-ike^{ikz}\hat{\rho} \right) = -\frac{ika\,B_0}{\rho} e^{i(kz-\omega t)} \hat{\rho}.$$

$$\frac{1}{c}\frac{\partial E}{\partial t} = \frac{\hat{\rho}}{c} \frac{E_0 a}{\rho} (-i\omega) e^{i(kz-\omega t)} = -\frac{i\omega}{c} a E_0 e^{i(kz-\omega t)} \hat{\rho}$$

Since $\omega = kc$ and $B_0 = E_0$,

$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}$ which is the 4th

Maxwell vacuum equation.

2.4.15 $\quad \vec{B} = \hat{\varphi} B_\varphi(\rho).$

$$(B \cdot \nabla)\vec{B} = B_\varphi(\rho) \frac{1}{\rho} \frac{\partial}{\partial \varphi}\left[\hat{\varphi} B_\varphi(\rho)\right] .$$

Now
$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi \qquad \text{so}$$

$$\frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{x}\cos\varphi - \hat{y}\sin\varphi = -\hat{\rho} .$$

So

$$(B \cdot \nabla)\vec{B} = B_\varphi(\rho) \frac{1}{\rho} B_\varphi(\rho)(-\hat{\rho}) = -\frac{\hat{\rho}}{\rho} B_\varphi^2(\rho) .$$